Section 65

Q2. Let \( f(z) = e^z \). We will obtain the Taylor expansion at \( z_0 = 1 \) in two different ways

(a) Note that \( f^{(n)}(z) = e^z \), thus \( f^{(n)}(1) = e \) for all \( n \). We can thus use Taylor’s theorem (section 62) for \( |z - 1| < \infty \) to obtain
\[
    f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (z - 1)^n = e \sum_{n=0}^{\infty} \frac{(z - 1)^n}{n!}.
\]

(b) We can write \( e^z = e^{1+(z-1)} = e \cdot e^{z-1} \). Using the Taylor expansion for the exponential function \( (e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}, \text{valid for } w \in \mathbb{C}) \),
\[
e^{z-1} = \sum_{n=0}^{\infty} \frac{(z - 1)^n}{n!}.
\]
Thus we obtain the required expression
\[
e^z = e \cdot e^{z-1} = e \sum_{n=0}^{\infty} \frac{(z - 1)^n}{n!}.
\]

Section 68

Q1. We know the Taylor series expansion for \( \sin z \) for \( |z| < \infty \)
\[
    \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.
\]
Replacing \( z \) by \( 1/z^2 \) in the above, we have the Laurent series expansion over the domain \( 0 < |z| < \infty \)
\[
    \sin \left( \frac{1}{z^2} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n+2}}.
\]
Multiplying the above by \( z^2 \) gives us
\[
    z^2 \sin \left( \frac{1}{z^2} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}}.
\]
Q2. We have the function

\[ f(z) = \frac{1}{1+z}. \]

Note that \( 1 < |z| < \infty \) implies \( 0 < |\frac{1}{z}| < 1 \). Hence we can use the geometric series expansion \( \frac{1}{1-w} = 1 + w + w^2 + \cdots (|w| < 1) \) to obtain

\[ \frac{1}{1-(\frac{1}{z})} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} \quad (0 < |\frac{1}{z}| < 1). \]

Using the above we obtain

\[ f(z) = \frac{1}{z(1-(1/z))} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}}. \]

Section 72
Q1. We have the Maclaurin series representation for \( |z| < 1 \):

\[ \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n. \]

By Theorem 2 in Section 71, we know that that within the circle of convergence, we can differentiate term by term to obtain

\[ \frac{1}{(z-1)^2} = \frac{d}{dz} \frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{d}{dz} z^n = \sum_{n=0}^{\infty} n z^{n-1} = \sum_{n=0}^{\infty} (n+1) z^n. \]

for \( |z| < 1 \). Differentiating the above Maclaurin series representation we obtain

\[ \frac{2}{(1-z)^3} = \frac{d}{dz} \frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} \frac{d}{dz} (n+1) z^n = \sum_{n=0}^{\infty} (n+1) \cdot n z^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2) z^n. \]

Section 73
Q1. Note that \( e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \) and for \( 0 < |z| < 1 \), we have

\[ \frac{1}{1+z^2} = 1 - z^2 + z^4 - \cdots. \]
The dots above represent the higher order terms. Thus by multiplying the above two power series expansion, we obtain
\[
e^{z} \left( \frac{1}{1 + z^2} \right) = \left( 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots \right) \left( 1 - z^2 + z^4 - \cdots \right)
\]
\[
= 1 + z + \left( \frac{z^2}{2} - z^2 \right) + \left( \frac{z^3}{6} - z \cdot z^2 \right) + \cdots
\]
\[
= 1 + z - \frac{z^2}{2} - \frac{5}{6} z^3 + \cdots
\]
Thus multiplying \( \frac{1}{z} \) to the above identity gives us
\[
\frac{e^{z}}{z(1 + z^2)} = \frac{1}{z} + 1 - \frac{z}{2} - \frac{5}{6} z^2 + \cdots
\]

Q2.

(a) Both the functions \( e^{z} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots \) and \( \sin z = z - \frac{z^3}{6} + \cdots \) are entire functions. Thus the product \( e^{z} \sin z \) is entire with Taylor expansion at origin starting with
\[
e^{z} \sin z = \left( 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots \right) \left( z - \frac{z^3}{6} + \cdots \right)
\]
\[
= z + z^2 + \left( \frac{z^2}{2} \cdot z - \frac{z^3}{6} \right) + \cdots
\]
\[
= z + z^2 + \frac{z^3}{3} + \cdots
\]

(b) Both the functions \( e^{z} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots \) and \( \frac{1}{1 + z} = 1 - z + z^2 - z^3 + \cdots \) are defined on \( |z| < 1 \). Thus the product \( \frac{e^{z}}{z+1} \) is defined on \( |z| < 1 \) with Taylor expansion at origin starting with
\[
\frac{e^{z}}{z+1} = \left( 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots \right) \left( 1 - z + z^2 - z^3 + \cdots \right)
\]
\[
= 1 + (z - z) + \left( \frac{z^2}{2} - z \cdot z + z^2 \right) + \left( \frac{z^3}{6} - \frac{z^2}{2} \cdot z + z \cdot z^2 - z^3 \right) + \cdots
\]
\[
= z + \frac{z^2}{2} - \frac{z^3}{3} + \cdots
\]

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