Final Exam Practice Problems

Midterm II will cover sections 91-94, 96-100, and 112, 113, 115. In terms of lecture notes, the final exam will cover all lecture notes since "May 2nd-May 4th, Chapter 7, part V" (included).

1. Compute the following integrals.

   (a) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}(x+1)}dx.$

   (b) $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}.$

2. Let $p(z) = 2z^5 - 6z^2 + z - 1$.

   (a) Find the number of zeros (counting multiplicity) of $p(z)$ inside the unit circle $|z| = 1$.

   (b) Find the number of zeros (counting multiplicity) of $p(z)$ inside the unit circle $|z| = 2$.

   (c) Find the number of zeros (counting multiplicity) of $p(z)$ in the annulus $\{1 < |z| < 2\}$.

3. True or False.

   (a) If $f(z)$ is analytic at z_0, then $f(z)$ is conformal at z_0.

   (b) The inverse of a linear fractional transformation is still a linear fractional transformation.

   (c) The inversion map $w = \frac{1}{z}$ sends lines to lines.

   (d) Let D be a domain. Any harmonic function u on D always has a harmonc conjugate.

   (e) Suppose u is harmonic on $\mathbb{R}^2$, (i.e., u has continuous second order partial derivatives and satisfies the Laplace equation on $\mathbb{R}^2$). Then u is a smooth function on $\mathbb{R}^2$, (i.e., its partial derivatives of all orders are continuous on $\mathbb{R}^2$).

   (f) $T(z) = z$ is the only linear fractional transformation that sends $0, i, 2$ to $0, i, 2$ respectively.

   (g) Let $T(z) = \frac{az+b}{cz+d}$ be a linear fractional transformation. Assume $c \neq 0$. Then T is conformal at $z_0 \in \mathbb{C}$ for every $z_0 \neq -\frac{d}{c}$.

4. In each part, determine the angle of rotation and scale factor for the following conformal maps at z_0.

   (a) $f(z) = \frac{1}{z}$ at $z_0 = i$.

   (b) $f(z) = e^z$ at $z_0 = \pi - i\pi$.

5. Consider the function $u(x, y) = e^{-3x} \cos(3y)$.

   (a) Show $u(x, y)$ is harmonic on the complex plane $\mathbb{C}$.

   (b) Find the harmonic conjugate $v(x, y)$ of $u(x, y)$ on $\mathbb{C}$, such that $v(0, 0) = 1$.

   (c). Find the analytic function $f(z)$ on $\mathbb{C}$ such that $f = u + iv$.

   (d). Find the angle between the level curves $u(x, y) = 1$ and $v(x, y) = 1$ at the intersection point $(0, 0)$.

6. Find the linear fractional transformation that maps the points $-1, 0, 1$ to $-i, 1, i$ respectively.

7. Find the linear fractional transformation that maps the points $\infty, i, 0$ to $0, i, \infty$ respectively. What is the image of the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ under this transformation?