Midterm I Practice Problems

Midterm I will cover sections 58, 60-62, 64-66, 68, 71-76, 78-83, and then pages 259-261 of section 85. That is, the exam will cover everything taught up to the lecture of Friday, 04/19 (included).

Tips:
You need to memorize:

- Statements of Liouville's Theorem and the Fundamental Theorem of Algebra
- The following Taylor series:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n; |z| < 1;$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}; z \in \mathbb{C};$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}; z \in \mathbb{C};$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}; z \in \mathbb{C};$$

- Definitions of $\int_{-\infty}^{\infty} f(x)dx$ and P.V. $\int_{-\infty}^{\infty} f(x)dx$, and their differences.

1. Let f be an entire function and assume $|f(z)| \geq 2$ for all $z \in \mathbb{C}$. Prove f is constant. (Hint: Consider a new function $g = \frac{1}{f}$, and apply Liouville's theorem).

2. (1) Find the expansion of $\frac{1}{2-z}$ on $\{|z| < 2\}$. Is it a Taylor series or a Laurent series?

   (2). Find the expansion of $\frac{1}{2-z}$ on $\{|z| > 2\}$. Is it a Taylor series or a Laurent series?

3. Let $f(z) = \frac{e^z}{z^2+1}$. Find $f^{(3)}(0)$ and $f^{(4)}(0)$.

4. In each part, for the given $f(z)$, classify the type of singularity, removable singularity, pole (if it is a pole, you need to find the order) or essential singularity, at z_0. And compute the residue $\operatorname*{Res}_{z=z_0} f(z)$.

   (a). $f(z) = z\cos(\frac{1}{z})$ and $z_0 = 0$.

   (b). $f(z) = \dfrac{\sin z}{z(z-\frac{\pi}{2})^2}$ and $z_0 = \frac{\pi}{2}$.

   (c). $f(z) = \frac{e^{2z}-1}{z^4}$ and $z_0 = 0$.

5. Evaluate the integral

$$\int_C \frac{\sin z}{z(z-\frac{\pi}{2})^2} dz$$

when C is the circle $|z-2| = 1$ with positive orientation.