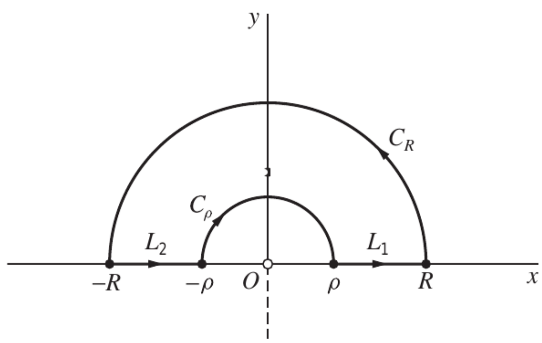
The exam starts here: The exam has 46 points including the bonus points, but we will grade it out of 45 points. That means we will add up all the points that you get, but the maximum total points you can have is 45.

Note: **When you apply a theorem or lemma, you need to write the name of the theorem or lemma (no need to write its statement).**

(8) **1.** For each of the following integrals, sketch a contour that can be used to evaluate the integral by residues, and give the analytic function $f(z)$ that you would integrate. If the function $f(z)$ involves log function, specify which branch you take. **You do not need to compute out the integral**.

1 (a). $\displaystyle\int_0^\infty \frac{\ln x}{x^4+x^2+1}dx.$

Ans. We shall consider the function $f(z) = \frac{\log z}{z^4+z^2+1}$, where we pick the branch of logarithm given by $\log z = \ln r + i\theta$ where $-\pi/2 < \theta < 3\pi/2$. The contour we shall consider is

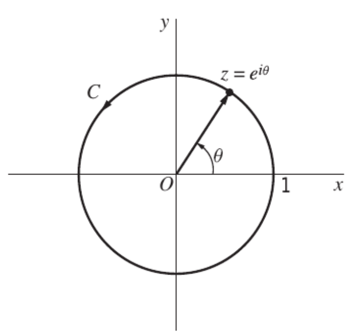


1 (b). $\displaystyle\int_0^{2\pi} \frac{1}{5+4\cos\theta}d\theta.$

Ans. We shall consider the function

$$f(z) = \frac{1}{5+4\left(\frac{z+z^{-1}}{2}\right)}\cdot\frac{1}{zi} = \frac{-i}{2z^2+5z+2}.$$

The contour we shall consider is



(5) **2 (a).** Let $R > 1$, and let C_R denote the upper semicircle $z(t) = Re^{it}, 0 \le t \le \pi$ (positively oriented). Compute the limit $\lim_{R\to\infty} \int_{C_R} \frac{ze^{iz}}{z^2+z+1} dz$. **Show your work**.

Ans. Using triangle inequality for $|z| = R > 2$ (this implies $|z+1| > |z| - 1 > 1$) we obtain

$$|z^2 + z + 1| \ge ||z^2 + z| - 1| = R|z+1| - 1 \ge R(R-1) - 1 = R^2 - R - 1.$$

Consider the function $f(z) := \frac{z}{z^2+z+1}$, then for $R > 2$

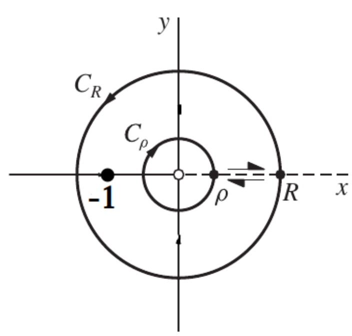
$$|f(z)| = \frac{|z|}{|z^2+z+1|} \le \frac{R}{R^2 - R - 1} =: M_R.$$

Note that $\lim_{R\to\infty} M_R = 0$. We apply Jordan's Lemma to obtain

$$\lim_{R\to\infty} \int_{C_R} f(z) e^{iz} dz = 0.$$

(11) **2 (b).** Use complex integral to evaluate $\int_0^\infty \frac{1}{\sqrt{x}(x+1)} dx$. **Show your work**.

Ans. Let $f(z) = \frac{z^{-1/2}}{z+1}$ where $z^{-1/2} = e^{(-1/2)\log z}$ where we pick the branch of logarithm given by $\log z = \ln r + i\theta$ and $0 < \theta < 2\pi$.



Consider the contour in Figure 3 (copies from the book fig 109). The two arrows between ρ and R refers to the upper edge (denoted as L_u) and lower edge (denoted as L_ℓ) given by $\theta = 0$ and $\theta = 2\pi$. Note that the orientation of L_ℓ is in opposite direction.
By Cauchy-Goursat theorem, we obtain

$$\int_{C_R} f(z)dz + \int_{L_u} f(z)dz + \int_{C_\rho} f(z)dz + \int_{L_\ell} f(z)dz = 2\pi i \operatorname*{Res}_{z=-1} f(z). \quad (1) \quad \boxed{\text{eq3}}$$

Let $R > 1$ and $\rho < 1$. We evaluate each of the above integrals:

- Integral over C_R: Note that we have the following inequality for $z \in C_R$

$$|f(z)| \le \frac{1}{\sqrt{R}(R-1)} = M_R$$

Hence

$$\left| \int_{C_R} f(z)dz \right| \leq M_R \cdot 2\pi R = \frac{2\pi R^{1/2}}{R-1}.$$

Thus, we obtain

$$\lim_{R\to\infty} \int_{C_R} f(z)dz = 0.$$

- Integral over C_ρ: Note that we have the following inequality for $z \in C_\rho$

$$|f(z)| \leq \frac{1}{\sqrt{\rho}(1-\rho)} = M_\rho$$

Hence $\left| \int_{C_\rho} f(z)dz \right| \leq M_\rho \cdot 2\pi\rho = \frac{2\pi\rho^{1/2}}{1-\rho}$. Thus, we obtain

$$\lim_{\rho\to\infty} \int_{C_\rho} f(z)dz = 0.$$

- Integral over L_u and L_ℓ: We use the parameterization $z(x) = x$ ($\rho \leq x \leq R$) for both L_u and for $-L_\ell$ to obtain

$$\begin{aligned}
\int_{L_u} f(z)dz - \int_{-L_\ell} f(z)dz &= \int_\rho^R \frac{e^{-\frac{1}{2}(\ln x)}}{x+1}dx - \int_\rho^R \frac{e^{-\frac{1}{2}(\ln x + i2\pi)}}{x+1}dx \\
&= \int_\rho^R \frac{1}{\sqrt{x}(x+1)}dx - \int_\rho^R \frac{e^{i\pi}}{\sqrt{x}(x+1)}dx \\
&= 2\int_\rho^R \frac{1}{\sqrt{x}(x+1)}dx.
\end{aligned}$$

- Residue at -1: Note that $f(z) = \frac{\phi(z)}{z+1}$ where $\phi(z) = z^{-1/2}$. Since $\phi(-1) \neq 0$, $f(z)$ has a simple pole at i and its residue equals

$$\underset{z=i}{Res} f(z) = \phi(-1) = e^{-\frac{1}{2}(\ln|-1| + i\pi)} = e^{-i\pi/2} = -i.$$

Taking the limit $\rho \to 0$ and $R \to \infty$ in (1) and using the calculations above, we obtain

$$2\int_0^\infty \frac{1}{\sqrt{x}(x^2+1)}dx = 2\pi i \cdot (-i).$$

which implies

$$\int_0^\infty \frac{1}{\sqrt{x}(x^2+1)}dx = \pi.$$

3. Let $p(z) = 2z^5 - 6z^2 + z - 1$.

(5) **3 (a).** Find the number of zeros (counting multiplicity) of $p(z)$ inside the unit circle $|z| = 1$.
Ans. Consider the function $f(z) = -6z^2$ and $g(z) = 2z^5 + z - 1$. By triangle inequality, for $|z| = 1$, we obtain

$$|g(z)| \leq |2z^5| + |z| + |1| = 2 + 1 + 1 = 4.$$

Since $|f(z)| = |6z^2| = 6$, we obtain $|f(z)| > |g(z)|$. By Rouche's theorem, the number of zeros of $f(z) + g(z) = p(z)$ and $f(z)$, inside the unit circle $|z| = 1$, are equal (counted with multiplicity).

Note that $f(z)$ has a zeros at $z = 0$ of order 2. Hence number of zeros of $p(z)$ **equals 2** as well.

(5) **3 (b).** Find the number of zeros (counting multiplicity) of $p(z)$ inside the unit circle $|z| = 2$.
Ans. Consider the functions $f(z) = 2z^5$ and $g(z) = -6z^2 + z - 1$. By triangle inequality, for $|z| = 2$, we obtain

$$|g(z)| \leq |6z^2| + |z| + |1| = 24 + 2 + 1 = 27.$$

Since $|f(z)| = |2z^5| = 32$, we obtain $|f(z)| > |g(z)|$. By Rouche's theorem, the number of zeros of $f(z) + g(z) = p(z)$ and $f(z)$, inside the circle $|z| = 2$, are equal (counted with multiplicity).

Note that $f(z)$ has a zeros at $z = 0$ of order 5. Hence number of zeros of $p(z)$ **equals 5** as well.

(3) **3 (c).** Find the number of zeros (counting multiplicity) of $p(z)$ in the annulus $\{1 < |z| < 2\}$.

Ans. Note that there are no zeros on the $|z| = 1$: Consider the function in part (a), $f(z) = -6z^2$ and $g(z) = 2z^5 + z - 1$. We showed that for $|z| = 1$, $|f(z)| > |g(z)|$, thus

$$|p(z)| = |f(z) + g(z)| \geq |f(z)| - |g(z)| > 0.$$

In particular, $p(z) \neq 0$ for any $|z| = 1$.
Thus the number of zeros of $p(z)$ in the annulus $1 < |z| < |2|$ equals $5 - 2 = 3$.

(8) **4.** Let $f(z) = \dfrac{(2z+i)^{100}(2z^5-6z^2+z-1)}{(2z-1)^2(z-2)^9}$. Let C denote the unit circle $z(t) = e^{it}, t \in [0, 2\pi]$, oriented positively, and Γ denote the image of C in the w-plane under the map $w = f(z)$. Compute the winding number of Γ around 0 in the w-plane, and then compute $\Delta_C(\arg f)$. **Show your work. You can use your answer of problem 3.**

Ans. The winding number of Γ around zero in w-plane is given by

$$\frac{1}{2\pi}\Delta_C \arg f(z).$$

Note that $f(z)$ has

- Zero at $i/2$: **order 100**; inside C
- Pole at $1/2$: **order 2**; inside C
- Pole at 2 : order 9; outside C
- For $p(z) = 2z^5 - 6z^2 + z - 1$, it has no poles, and **exactly 2 zeros** inside C.

Hence $Z = 100 + 2$ and $P = 2$, where Z is the number of zeros (counted with multiplicity) and P is the number of poles (counted with multiplicity) inside C. Hence, by Argument principle (Theorem in section 86), the winding number equals

$$\frac{1}{2\pi}\Delta_C \arg f(z) = Z - P = 100.$$

Therefore $\Delta_C \arg f(z) = 200\pi$.

The following question is optional and you get up to 1 bonus point if you solve it.

(1) **5.** Do you have any suggestions to improve the quality of our future lectures? Whatever suggestions you make, you will get the 1 point for this problem. Your suggestions are highly appreciated.

Thank you very much for your comments/suggestions!

The exam ends here.