Math 120B Complex Analysis Midterm I April 22nd, 2022

Name (PRINT):________________________ PID:____________

Signature: ________________________

Important Instructions: No books, notes, cell phones, or any other electronic devices may be used during the exam. Do not start the exam until instructed to do so. Except where otherwise indicated, you need to justify your answer. No credit will be given for unsupported answer, even if correct. You cannot use any result in your homework unless you reprove it.

Problem	Points	Score
#1	13	
#2	10	
#3	10	
#4	12	
#5	2	
Total	45	

The exam starts here: The exam has 47 points including the bonus points, but we will grade it out of 45 points. That means we will add up all the points that you get, but the maximum total points you can have is 45.

(5) **1 (a).** State the fundamental theorem of algebra.

Answer: Every polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ $(a_n \neq 0)$ with degree $n \geq 1$ has at least one zero in $\mathbb{C}$ (i.e a point $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$).

(8) **1 (b).** Let $f(z) = \frac{e^z}{1-z^2}$. **Use Taylor series** to find $f^{(3)}(0)$. Justify your answer.
Answer: We have the following Taylor expansion for e^z and $\frac{1}{1-z^2}$ (for $|z| < 1$) at $z_0 = 0$:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$
$$\frac{1}{1-z^2} = 1 + z^2 + z^4 + \cdots .$$

Therefore, the first few terms of the Taylor expansion of their product (for $|z| < 1$) is given by

$$f(z) = \frac{e^z}{1-z^2} = 1 + z + \left(\frac{z^2}{2!} + z^2\right) + \left(\frac{z^3}{3!} + z \cdot z^2\right) + \cdots$$
$$= 1 + z + \frac{3}{2}z^2 + \frac{7}{6}z^3 + \cdots$$

Recall that the Taylor series at $z_0 = 0$ equals

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

Comparing the coefficient z^3 in the above two expansion, we obtain

$$f^{(3)}(0) = 3! \cdot \frac{7}{6} = 7.$$

(10) **2.** In each part, for the given $f(z)$, classify the type of singularity, removable singularity, pole (**if it is a pole, you need to find the order**) or essential singularity, at z_0. **And compute the residue** $\operatorname*{\mathbf{Res}}_{z=z_0} f(z)$. Justify your answer.

(4) **2 (a).** $f(z) = ze^{\frac{1}{z}}$ and $z_0 = 0$.

Answer: The Laurent series of the function $f(z)$ at $z_0 = 0$ is given by

$$\begin{aligned} f(z) &= z \cdot \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} \\ &= z\left(1 + \frac{1}{z} + \frac{1}{z^2 2!} + \frac{1}{z^3 3!} + \cdots\right) \\ &= z + 1 + \left(\frac{1}{2!z} + \frac{1}{3!z^2} + \cdots\right) \\ &= z + 1 + \sum_{n=1}^{\infty} \frac{1}{z^n(n+1)!}. \end{aligned}$$

Since the principal part of the Larent expansion $\sum_{n=1}^{\infty} \frac{1}{z^n(n+1)!}$ has infinitely many non-zero terms, we conclude that $f(z)$ has an **essential singularity** at $z_0 = 0$.
The residue equals the coefficient of $\frac{1}{(z-z_0)}$ in the Laurent series expansion of $f(z)$ at z_0. Thus

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{2}.$$

(6) **2 (b).** $f(z) = \dfrac{\sin z + \cos z}{z^2}$ and $z_0 = 0$.

Answer: Let $\phi(z) = \sin z + \cos z$. Observe that $\phi(z)$ is analytic and non-zero (since $\phi(0) = 1$) at $z_0 = 0$. Using the Theorem in section 80 (in the textbook),

$$f(z) = \frac{\phi(z)}{z^2}$$

has a pole of order 2 at $z_0 = 0$.
Moreover, the residue at $z_0 = 0$ is given by

$$\begin{aligned} \operatorname*{Res}_{z=0} f(z) &= \frac{\phi^{(1)}(0)}{1!} \\ &= \cos 0 - \sin 0 \\ &= 1 \end{aligned}$$

(10) **3.** Determine whether the following statements are true or false. Write "True" or "False" in the parentheses. **You don't need to justify.**

(1). Let f be a real continuous function. If the principal value P.V. $\int_{-\infty}^{\infty} f(x)dx = 0$, then the full integral value $\int_{-\infty}^{\infty} f(x)dx = 0$. ( )
False : consider $f(x) = x$.

(2). If f and g both have a pole at z_0, then $f + g$ also has a pole at z_0. ( )
False : consider $f(z) = \frac{1}{z}$ and $g(z) = \frac{-1}{z}$ at $z_0 = 0$.

(3). If f has a removable singularity at z_0 and g has a pole at z_0, then $f + g$ has a pole at z_0. ( )
True Use the Theorem in section 80.

(4). Let ϕ be an analytic function at z_0 and let m be a positive integer. Then $\frac{\phi}{(z-z_0)^m}$ has a pole of order m at z_0. ( )
False : consider $\phi(z) = (z - z_0)^m$.

(5). If f has a zero of order 4 at 0, then $\frac{z+1}{f}$ has a pole of order 4 at 0. ( )
True : Use the Theorem in Section 80 and Theorem 1 in Section 82.

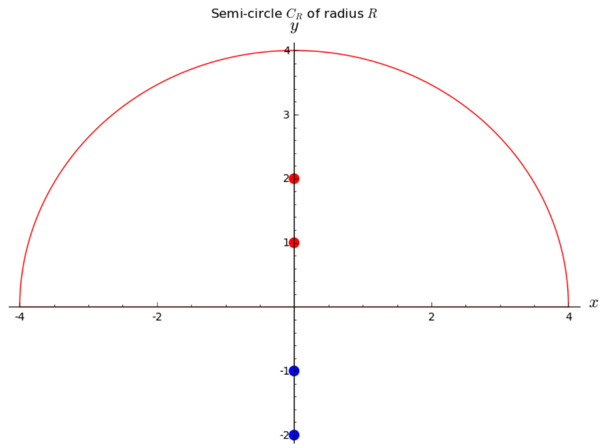

Figure 1: Contour integral

(12) **4.** Compute $\int_{-\infty}^{\infty} \frac{x^2+10}{(x^2+1)(x^2+4)} dx$ and $\int_{0}^{\infty} \frac{x^2+10}{(x^2+1)(x^2+4)} dx$. Show your work.

Answer: Consider the rational function $f(z) = \frac{z^2+10}{(z^2+1)(z^2+4)}$. Note that $f(z)$ has isolated singularities (poles) at $z_0 \in \{\pm i, \pm 2i\}$.
Let C_R be the semi-circle of radius R centered at origin (in the upper half plane) as shown in the Figure 1 (oriented in counter-clockwise direction). For $R > 2$, we can use Cauchy's Residue theorem to write

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \left[\operatorname*{Res}_{z_0=i} f(z) + \operatorname*{Res}_{z_0=2i} f(z) \right].$$

- Let $\phi(z) = \frac{z^2+10}{(z+i)(z^2+4)}$, then $f(z) = \frac{\phi(z)}{z-i}$. Hence, the residue is given by

$$\operatorname*{Res}_{z_0=i} f(z) = \phi(i) = \frac{i^2+10}{(i+i)(i^2+4)} = \frac{3}{2i}.$$

- Let $\phi(z) = \frac{z^2+10}{(z^2+1)(z+2i)}$, then $f(z) = \frac{\phi(z)}{z-2i}$. Hence, the residue is given by

$$\operatorname*{Res}_{z_0=2i} f(z) = \phi(2i) = \frac{4i^2+10}{(4i^2+1)(2i+2i)} = \frac{-1}{2i}.$$

Using the above residue calculation, we obtain

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \left[\frac{3}{2i} - \frac{1}{2i} \right] = 2\pi. \qquad (1)$$

- $\lim\limits_{R\to\infty} \int_{C_R} f(z)dz = 0$: For $R > 4$, we have the inequalities $|z^2+10| \leq |z^2| + 10 = R^2 + 10$, $|z^2+1| \geq ||z^2| - 1| = R^2 - 1$ and $|z^2+4| \geq ||z^2| - 4| = R^2 - 4$. Thus, we have

$$|f(z)| \leq M_R = \frac{(R^2+10)}{(R^2-1)(R^2-4)}.$$

The length of the semi-circle C_R equals πR. Hence

$$\int_{C_R} f(z)dz \leq M_R \cdot \pi R = \frac{\pi R(R^2+10)}{(R^2-1)(R^2-4)}$$

Taking the limit $R \to \infty$, we obtain

$$\lim_{R\to\infty} \int_{C_R} f(z)dz = \lim_{R\to\infty} \frac{\frac{\pi}{R}(1+\frac{10}{R^2})}{(1-\frac{1}{R^2})(1-\frac{4}{R^2})} = 0. \tag{2}$$

Since $f(x)$ is an **even function**, the Principal Value matches the full integral, that is,

$$2\int_0^\infty f(x)dx = \int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty} \int_{-R}^R f(x)dx.$$

Using (2) and (1) we obtain

$$\int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty} \int_{-R}^R f(x)dx = 2\pi - \lim_{R\to\infty} \int_{C_R} f(z)dz = 2\pi.$$

$$\int_0^\infty f(x)dx = \frac{1}{2}\int_{-\infty}^\infty f(x)dx = \pi.$$

The following question is optional and you get up to 2 bonus points if you solve it.

(2) **5.** Let f be an analytic function at z_0 and let m be a positive integer. Assume f has a zero of order m at z_0. Prove $\frac{f'}{f}$ has a pole at z_0. Find the order of this pole. Justify your answer.

Proof:
Using Theorem 1 in Section 82, we know that there exists a function $g(z)$, which is analytic and non-zero at z_0, such that
$$f(z) = (z - z_0)^m g(z).$$
Using product rule for differentiation, we obtain
$$\begin{aligned} f'(z) &= m(z - z_0)^{m-1} g(z) + (z - z_0)^m g'(z) \\ &= (z - z_0)^{m-1}(mg(z) + (z - z_0)g'(z)). \end{aligned}$$
Note that the function $h(z) := mg(z) + (z - z_0)g'(z)$ is analytic and non-zero at z_0 (since $h(z_0) = mg(z_0) \neq 0$). Thus $f'(z) = (z - z_0)^{m-1}h(z)$ has a zero of order $m - 1$ at z_0. For some punctured disc around z_0, we can write
$$\frac{f'(z)}{f(z)} = \frac{(z - z_0)^{m-1}h(z)}{(z - z_0)^m g(z)} = \frac{\phi(z)}{(z - z_0)},$$
where $\phi(z) = \frac{h(z)}{g(z)}$ is analytic and non-zero at z_0. Hence the function $\frac{f'}{f}$ has **pole** of order 1.

The exam ends here.