3. Indented Paths

Example: Evaluate \( \int_0^\infty \frac{\sin x}{x} \, dx \).

- Recall that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \), so the integrand does not blow up at \( x=0 \).
- Still not clear by inspection if the integral exists.
- Recall that \( \sin x \) grows rapidly as \( \text{Im} \, z \) gets large, so again we want to use the trick of writing \( \int_0^\infty \frac{\sin x}{x} \, dx = \text{Im} \int_0^\infty \frac{e^{iz}}{z} \, dx \).
- Jordan’s Lemma allows us to deal with \( \int_{C_R} \frac{e^{iz}}{z} \, dz \) (i.e. show that this \( \to 0 \) as \( R \to \infty \)) but we also need to deal with the fact that \( \frac{e^{iz}}{z} \) has a simple pole at \( z=0 \) (we would usually integrate over \( [-R,R] \) so this poses a problem!).

Consider \( f(z) = \frac{e^{iz}}{z} \). It is analytic on \( C-\{0\} \) & has a simple pole at \( 0 \).

Go around the pole!

We will send \( R \to \infty \) and \( \rho \to 0 \).
\[ \Gamma_{R,p} = [-R, -p] + C_p + \lbrack p, R \rbrack + C_R \]
\[ L_2 \quad \text{clockwise} \quad L_1 \quad \text{counterclockwise} \]
\[ (R > p > 0) \]

Cauchy–Goursat (no singularity inside \( \Gamma_{R,p} \)):
\[ \int_{-L_2} \frac{e^{iz}}{z} \, dz + \int_{-L_1} \frac{e^{iz}}{z} \, dz + \int_{L_p} \frac{e^{iz}}{z} \, dz + \int_{C_R} \frac{e^{iz}}{z} \, dz = 0 \]
\[ [\varepsilon, -p] \]

Now
\[ \int_{L_1} \frac{e^{iz}}{z} \, dz = \int_{p}^{R} \frac{e^{it}}{t} \, dt \]
\[ \int_{L_2} \frac{e^{iz}}{z} \, dz = -\int_{-L_2} \frac{e^{iz}}{z} \, dz \]
\[ = -\int_{-p}^{R} \frac{e^{-it}}{t} \cdot (-dt) \]
\[ = -\int_{p}^{R} \frac{e^{-it}}{t} \, dt. \]

\[ \Rightarrow \int_{L_1} \frac{e^{iz}}{z} \, dz + \int_{L_2} \frac{e^{iz}}{z} \, dz = \int_{p}^{R} \frac{e^{it} - e^{-it}}{t} \, dt \]
\[ = 2i \int_{p}^{R} \frac{\sin t}{t} \, dt. \]

Jordan's Lemma: Since on \( C_R \)
\[ \left| \frac{1}{z} \right| = \frac{1}{R} \to 0 \text{ as } R \to \infty \]

by Jordan's Lemma
\[ \lim_{R \to \infty} \int_{C_R} \frac{e^{iz}}{z} \, dz = 0. \]
It remains to compute \( \lim_{\rho \to 0} \int_{C_\rho} \frac{e^{iz}}{z} \, dz \).

Note that \( \frac{e^{iz}}{z} = \frac{1}{z} + \frac{iz}{2} + \frac{(iz)^2}{2^2 z} + \cdots \). We write

\[
\int_{C_\rho} \frac{e^{iz}}{z} \, dz = \int_{C_\rho} \frac{e^{iz} - 1}{z} \, dz + \int_{C_\rho} \frac{1}{z} \, dz.
\]

\[
\int_{C_\rho} \frac{1}{z} \, dz = -\int_{C_\rho} \frac{dz}{z} = \int_{0}^{\infty} i e^{it} dt = -i\pi.
\]

On the other hand, \( \frac{e^{iz} - 1}{z} \) has a removable singularity at \( z = 0 \) (so can be extended to an entire function). It follows that \( \frac{e^{iz} - 1}{z} \) is bounded on \( |z| \leq 1 \): there is \( M > 0 \) s.t.

\[
\left| \frac{e^{iz} - 1}{z} \right| \leq M \quad \text{for} \quad |z| \leq 1.
\]

\[
\left| \int_{C_\rho} \frac{e^{iz} - 1}{z} \, dz \right| \leq M \pi \rho \to 0 \quad \text{as} \quad \rho \to 0.
\]

\[
\lim_{\rho \to 0} \int_{C_\rho} \frac{e^{iz}}{z} \, dz = -i\pi.
\]

So, letting \( R \to \infty, \rho \to 0 \), we get

\[
2i \int_{0}^{\infty} \frac{\sin x}{x} \, dx - i\pi = 0.
\]

\[
\int_{0}^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2}.
\]
Theorem:
\[ f(z) \text{ analytic in } 0 < |z - z_0| < R_2, \]
x_0 a simple pole of \( f \),
\( C_\rho \) : upper semicircle \( |z - x_0| = \rho \), clockwise.
Then
\[ \lim_{\rho \to 0} \int_{C_\rho} f(z) \, dz = -\pi i \cdot \text{Res}_{z=x_0} f(z). \]

Proof: Let \( B_0 = \text{Res}_{z=x_0} f(z) \).
Since \( x_0 \) is a simple pole of \( f(z) \), \( \frac{B_0}{z-x_0} \)
is the principal part of \( f(z) \) centered at \( x_0 \).

\[ f(z) = (f(z) - \frac{B_0}{z-x_0}) + \frac{B_0}{z-x_0}. \]
\( f(z) - \frac{B_0}{z-x_0} \) is analytic in \( |z - x_0| < R_2 \),
and therefore bounded (say by \( M \)) there.

\[ \left| \int_{C_\rho} (f(z) - \frac{B_0}{z-x_0}) \, dz \right| \leq M \pi \rho \to 0 \quad \text{as } \rho \to 0. \]

Direct computation gives \( \int_{C_\rho} \frac{B_0}{z-x_0} \, dz = -i\pi B_0 \).

\[ \lim_{\rho \to 0} \int_{C_\rho} f(z) \, dz = -i\pi B_0. \]