19. Residues at Poles

The 3 types of isolated singularities:

- isolated singularity of \( f(z) \),

\[
f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n},
\]

\( 0 < |z-z_0| < R_2 \).

3 possibilities:

(a) **Removable singularity**: principal part = 0, i.e. \( b_n = 0 \) for all \( n = 1, 2, 3, \ldots \).

(b) **Essential singularity**: infinitely many of the \( b_n \) are nonzero.

(c) **Pole**: principal part doesn't vanish, but only finitely many of the \( b_n \) are nonzero.

\[
\sum_{n=1}^{\infty} b_n (z-z_0)^n = \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}
\]

\( b_m \neq 0 \), \( b_j = 0 \) \( \forall j > m \).

\( \Rightarrow \) "pole of order \( m \)"

**Pole of order 1**: "simple pole"
Examples:

1. \( f(z) = \frac{e^z - 1}{z} \), \( z = 0 \) is a removable singularity.

\[
\begin{align*}
\frac{\hat{f}(z)}{z} & = \frac{1 + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots - 1}{z} \\
& = 1 + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} \\
& \text{for } z \in \mathbb{C} - \{0\}.
\end{align*}
\]

\[
\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}
\]

converges for all \( z \in \mathbb{C} \) and defines an entire function.

2. \( e^{\frac{1}{z}} = 1 + \frac{1}{2} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \frac{1}{4!z^4} + \cdots \)

has an essential singularity at \( z = 0 \).

3. \( \frac{e^z - 1}{z^2} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z^n}{(n+2)!} \)

\((m=1)\)

has a simple pole at \( z = 0 \).

4. \( \frac{z+i}{z^2+1} = \frac{z+i}{(z+i)(z-i)} \) has a removable singularity at \( z = -i \) and a simple pole at \( z = i \).

5. \( \frac{e^z}{(z-2)^5} \) \( \sim \) pole of order 5 at \( z = 2 \).
Residues at Poles:

Let \( g(z) \) be analytic at 0 with \( g(0) \neq 0 \), then \( \frac{g(z)}{z^m} \) has a pole of order \( m \) at 0 (here \( m \in \{1, 2, 3, \ldots \} \)).

Write \( g(z) = \sum_{n=0}^{\infty} a_n z^n , \ z \) near \( 0 \).

\[ \frac{g(z)}{z^m} = \frac{a_0}{z^m} + \frac{a_1}{z^{m-1}} + \ldots + \frac{a_{m-1}}{z} + a_m + a_{m+1}z + \ldots \]

The principal part

\[ \lim_{z \to 0} \frac{g(z)}{z^m} = a_{m-1} = \frac{g^{(m-1)}(0)}{(m-1)!}. \]

Note: for \( m=1 \), \( \frac{g(z)}{z} = \frac{a_0}{z} + a_1 + a_2z + \ldots \)

and \( \lim_{z \to 0} \frac{g(z)}{z} = a_0 = g(0) \).

Theorem:

An isolated singularity \( z_0 \) of \( f(z) \) is a pole of order \( m \)

\[ f(z) = \frac{\phi(z)}{(z-z_0)^m} , \phi \text{ analytic at } z_0, \text{ with } \phi(z_0) \neq 0. \]

In this case \( \lim_{z \to z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \).
Proof: It is easy to expand \( f(z) \) about \( z_0 \).

\[ f(z) = \frac{b_m}{(z-z_0)^m} + \cdots + \frac{b_1}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n \]

with \( b_m \neq 0 \).

\[ \Rightarrow \text{set } \phi(z) = (z-z_0)^m f(z) \]

\[ = b_m + b_m (z-z_0) + \cdots + b_1 (z-z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} \]

and define \( \phi(z_0) = b_m \). We have to do this since \( f(z) \) is not defined at \( z_0 \).

\[ \Rightarrow \phi(z) \text{ is analytic at } z_0, \]

\[ \phi(z_0) = b_m \neq 0 \text{ and } f(z) = \frac{\phi(z)}{(z-z_0)^m}. \]

\[ \Box \]

Examples:

1. \( f(z) = \frac{z^2+1}{z^2+4} \), isolated singularities at \( z = \pm 2i \).

\[ f(z) = \frac{z^2+1}{(z-2i)(z+2i)} = \frac{\phi(z)}{z-2i}, \]

where \( \phi(z) = \frac{z^2+1}{z+2i} \) is analytic at \( 2i \)

\[ \phi(2i) = \frac{-4+1}{2i+2i} = \frac{-3}{4i} = \frac{-3i}{4} \neq 0. \]

\[ \Rightarrow \text{Res}_{z=2i} f(z) = \frac{3i}{4}. \]
(2) $f(z) = \frac{z^4 + 1}{(z+i)^3}$, $z = -i$ is a pole of order 3.

(since $(-i)^4 + 1 \neq 0$)

\[ f(z) = \frac{\phi(z)}{(z+i)^3}, \quad \phi(z) = z^4 + 1, \]
\[ \phi(-i) = (-i)^4 + 1 = 1 + 1 = 2 \neq 0. \]

\[ \Rightarrow \text{Res } f(z) = \frac{\phi''(-i)}{2!} \]
\[ \phi'(z) = 4z^3, \quad \phi''(z) = 12z^2, \]
\[ \phi''(-i) = 12(-i)^2 = -12. \]

\[ \Rightarrow \text{Res } f(z) = -6. \]

(3) $f(z) = \frac{\log z}{(z-i)^2} = \frac{\phi(z)}{(z-i)^2}$

$\phi(i) = \log(i) = i \cdot \frac{\pi}{2} \neq 0.$

\[ \Rightarrow f(z) \text{ has a pole of order 2 at } z = i. \]

\[ \Rightarrow \text{Res } f(z) = \phi'(i) = \frac{1}{i} \bigg|_{z=i} = -i. \]

(4) $f(z) = \frac{\sin z}{z^4} = \frac{1}{z^4} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$

\[ = \frac{1}{z^4} \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right) \]
\[ = \frac{1}{2} \left( -\frac{1}{6z} + \frac{z}{5!} + \cdots \right) \]

\[ \Rightarrow \text{Res } f(z) = -\frac{1}{6}. \]
Zeros of analytic functions:

Defn: If \( f(z) \) is analytic at \( z_0 \), \( f(z) \) has a zero of order \( m \) at \( z_0 \)

\[
\Downarrow
\]

\( f(z_0) = f'(z_0) = \ldots = f^{(m-1)}(z_0) = 0 \)
\( \text{and}\ f^{(m)}(z_0) \neq 0. \)

Examples:

1. \( f(z) = z^4 \) has a zero of order 4 at 0.

2. \( \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \) has a zero of order 1 at 0.
   
   (All the zeros of \( \sin z \) have order 1.)

3. \( e^z - 1 - z \) has a zero of order 2 at 0.

Theorem: Let \( f(z) \) be analytic at \( z_0 \).

(i) \( f(z) \) has a zero of order \( m \) at \( z_0 \)

\[
\Downarrow
\]

(ii) \( f(z) = (z-z_0)^m \, g(z), \)
\( g(z) \) analytic at \( z_0 \), \( g(z_0) \neq 0. \)
Proof: Follows from the Taylor expansion of $f$ at $z_0$.

(i) $\Rightarrow f(z) = \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \cdots$

$\Rightarrow f(z) = (z-z_0)^m g(z)$

where $g(z) = \sum_{n=0}^{\infty} \frac{f^{(n+m)}(z_0)}{(n+m)!} z^n$, $z$ near $z_0$.

$\left( g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0 \right)$

(ii) $\Rightarrow f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$

by direct calculation, and similarly

$f^{(m)}(z_0) = m! g(z_0)$.

$\Rightarrow (i)$ □

Remark:

Note that the zeros of an analytic function must be isolated (since otherwise one can argue that all the terms in the Taylor expansion vanish and the function is just zero).
Zeroes and Poles:

If \( z_0 \) is a zero of order \( m \) of \( f(z) \) then \( z_0 \) is a pole of order \( m \) of \( 1/f(z) \).

To see this, write \( f(z) = (z-z_0)^m g(z) \).

\[ g \text{ analytic at } z_0, \ g(z_0) \neq 0. \]

\[ \Rightarrow \frac{1}{f(z)} = \frac{1}{(z-z_0)^m g(z)} = \frac{\phi(z)}{(z-z_0)^m}, \]

\[ \phi(z) = \frac{1}{g(z)} \text{ analytic at } z_0, \]

\[ \phi(z_0) = \frac{1}{g(z_0)} 
eq 0. \]

\[ \Rightarrow \text{Res}_{z=z_0} \frac{1}{f(z)} = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \]

Example: \( f(z) = (z+z^2)^2 \), \( f(0) = 0. \)

\[ \Rightarrow f(z) = z^2(1+z)^2 \text{ so } z=0 \text{ is a zero of order } 2. \]

\[ \Rightarrow h(z) = \frac{1}{f(z)} \text{ has a pole of order } 2. \]

\[ h(z) = \frac{1}{z^2(1+z)^2} = \frac{\phi(z)}{z^2}, \ \phi(z) = \frac{1}{(1+z)^2} \]

\( \phi \text{ analytic at } 0, \ \phi(0) = 1 \neq 0. \)

\[ \Rightarrow \text{Res}_{z=0} h(z) = \phi'(0) = \frac{-2}{(1+0)^3} \bigg|_{z=0} = -2. \]
Slightly more generally we have:

**Theorem:** If \( p(z) \) & \( q(z) \) are analytic functions such that

1. \( q(z) \) has a zero of order \( m \) at \( z_0 \);
2. \( p(z_0) \neq 0 \);

then \( \frac{p(z)}{q(z)} \) has a pole of order \( m \) at \( z_0 \).

*(Proof follows from the above discussion.)*

In the case \( m=1 \) there is a simple formula for \( \text{Res}_{z=z_0} \frac{p(z)}{q(z)} \).

**Theorem:** \( p(z), q(z) \) analytic.

If \( q(z) \) has a zero of order \( 1 \) at \( z_0 \) and \( p(z_0) \neq 0 \), then \( \frac{p(z)}{q(z)} \) has a simple pole at \( z_0 \) and

\[
\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.
\]

**Proof:** \( q(z) = (z-z_0)g(z) \), \( g \) analytic at \( z_0 \), \( g(z_0) \neq 0 \).

\[
\Rightarrow \frac{p(z)}{q(z)} = \frac{\phi(z)}{z-z_0}, \quad \phi(z) = \frac{p(z)}{g(z)}.
\]
\[ \operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \phi(z_0) = \frac{p(z_0)}{q'(z_0)}, \]

but \( q(z_0) = q'(z_0). \)

\[ \uparrow \quad q(z) = (z-z_0)q'(z_0) + \ldots = (z-z_0)q(z). \]

**Example:** \( f(z) = \cot z = \frac{\cos z}{\sin z}. \) \( \sin z \) has a zero of order 1 at 0 and \( \cos 0 = 1 \neq 0, \) so the same applies.

\[ \Rightarrow \operatorname{Res}_{z=0} \cot z = \frac{\cos 0}{\cos 0} = 1. \]

By the same argument

\[ \operatorname{Res}_{z=\pi} \cot z = 1 = \frac{\cos \pi}{\cos \pi} \]

for any \( n \in \mathbb{Z}. \)