§ 89. An indented path.

E.g. $\int_0^\infty \frac{\sin x}{x}\,dx$.

Is the improper integral convergent ?

$f(x) = \frac{\sin x}{x}$.

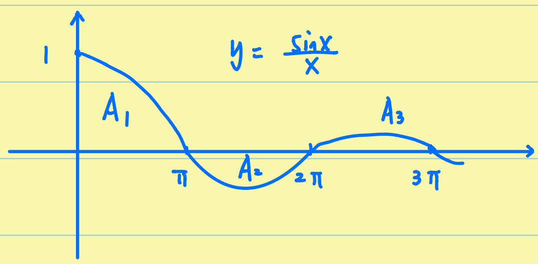
- $f(x)$ is continuous on $(0, \infty)$ but undefined at $x = 0$.

  Note $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

  By defining $f(0) = 1$,

  f is continuous on $[0, \infty)$.

  So $x = 0$ is not a problem.

- 

$$\int_0^\infty \frac{\sin x}{x}\,dx = \underbrace{A_1 - A_2 + A_3 - A_4 \cdots}_{\text{alternating series with } A_j \downarrow 0 \text{ as } j \to \infty.}$$

By the alternating series test, $\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent.

Now Let's evaluate $\int_0^\infty \frac{\sin x}{x}\,dx$.

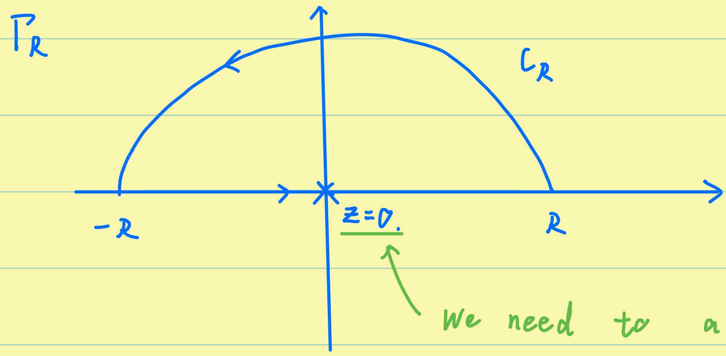
- Recall $|\sin z| \to \infty$ as $\operatorname{Im} z \to \infty$.

  We need to use the trick of writing

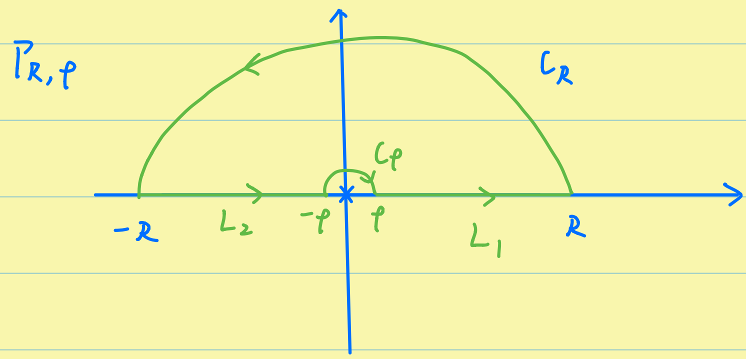
$$\int_0^\infty \frac{\sin x}{x}\,dx = \operatorname{Im} \int_0^\infty \frac{e^{ix}}{x}\,dx.$$

$f(z) = \dfrac{e^{iz}}{z}.$

f has a simple pole at $z = 0$.



We need to avoid this pole.



We will let $R \to \infty$ and $\rho \to 0$.

$$\Gamma_{R,\rho} = \underset{}{C_R} + \underset{[-R,-\rho]}{L_2} + \underset{\substack{\uparrow \\ \text{clockwise.}}}{C_\rho} + \underset{[\rho, R].}{L_1}$$

(*) $$\underset{①}{\int_{\Gamma_{R,\rho}} f(z)\,dz} = \underset{②}{\int_{C_R} f(z)\,dz} + \underset{③}{\int_{L_2} f(z)\,dz} + \underset{④}{\int_{C_\rho} f(z)\,dz} + \underset{⑤.}{\int_{L_1} f(z)\,dz.}$$

①. $\int_{\Gamma_{R,\rho}} f(z)\,dz = 0.$

No singularity inside $\Gamma_{R,\rho}$.

② $\int_{C_R} f(z)\,dz \qquad f(z) = \frac{e^{iz}}{z}.$

For $z \in C_R$,

$$\left|\frac{1}{z}\right| = \frac{1}{R} \longrightarrow 0 \quad \text{as } R \to \infty.$$

By Jordan's lemma,

$$\int_{C_R} \frac{e^{iz}}{z}\,dz \longrightarrow 0. \quad \text{as } R \to \infty.$$

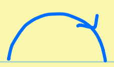
③ $\int_{[-R,-\rho]} \frac{e^{iz}}{z}\,dz = \int_{-R}^{-\rho} \frac{e^{ix}}{x}\,dx.$

⑤ $\int_{[\rho,R]} \frac{e^{iz}}{z}\,dz = \int_{\rho}^{R} \frac{e^{ix}}{x}\,dx.$

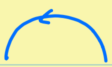
④ $\int_{C_\rho} \frac{e^{iz}}{z}\,dz.$

$$\frac{e^{iz}}{z} = \frac{1}{z}\left(1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} \cdots\right)$$

$$= \underbrace{\frac{1}{z}}_{\text{principal part.}} + \underbrace{i + \frac{i^2}{2!}z + \frac{i^3}{3!}z^2 + \cdots}_{\text{analytic part.}}$$


C_ρ.

$$\int_{C_\rho} \frac{e^{iz}}{z}\,dz = \int_{C_\rho} \frac{1}{z}\,dz + \int_{C_\rho} \frac{e^{iz}-1}{z}\,dz.$$


$-C_\rho$.

$z = \rho e^{i\theta}$

$\theta \in [0,\pi]$.

$$\int_{C_\rho} \frac{1}{z}\,dz = -\int_{-C_\rho} \frac{1}{z}\,dz = -\int_0^{\pi} \frac{1}{\rho e^{i\theta}}\,\rho \cdot i e^{i\theta}\,d\theta = -\pi i.$$

On the other hand, $\frac{e^{iz}-1}{z}$ has a removable singularity at $z=0$.

So $\frac{e^{iz}-1}{z}$ can be extended analytically across $z=0$.

It follows that $\frac{e^{iz}-1}{z}$ is bounded on $\{|z| \le 1\}$.

$\exists\ M>0$ s.t. $\left|\frac{e^{iz}-1}{z}\right| \le M$ for any $|z|\le 1$.

$\left| \int_{C_\rho} \frac{e^{iz}-1}{z} dz \right| \le M \cdot \pi\rho \to 0$ as $\rho \to 0$.

$\lim_{\rho\to 0} \int_{C_\rho} \frac{e^{iz}-1}{z} dz = 0.$

So $\lim_{\rho\to 0} \int_{C_\rho} \frac{e^{iz}}{z} dz = -\pi i.$

(*) $\int_{\Gamma_{R,\rho}} f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{C_\rho} f(z)\,dz + \int_{L_1} f(z)\,dz.$

$0 = \int_{C_R} f(z)\,dz + \int_{-R}^{-\rho} f(x)\,dx + \int_{C_\rho} f(z)\,dz + \int_{\rho}^{R} f(x)\,dx.$

Let $R\to\infty$ and $\rho\to 0$.

$0 = \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx - \pi i.$

$\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = i\pi.$

Take the imaginary parts.

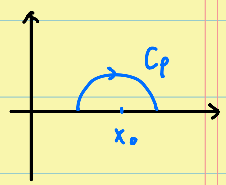
$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$

So $\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{1}{2}\pi.$

In general, we have the following theorem for indented paths.

Thm. $f(z)$ is analytic in $0 < |z - x_0| < R$.

x_0 is a simple pole of f.



C_ρ : upper semicircle $|z - x_0| = \rho$, clockwise orientation.

Then

$$\lim_{\rho \to 0} \int_{C_\rho} f(z)\, dz = -\pi i \cdot \operatorname*{Res}_{z = x_0} f(z).$$

proof. Let $B = \operatorname*{Res}_{z=x_0} f(z)$.

Since x_0 is a simple pole of f,

the principal part of f at x_0 is $\dfrac{B}{z - x_0}$.

$$f(z) = \underbrace{\frac{B}{z - x_0}}_{\text{principal}} + \underbrace{\left(f(z) - \frac{B}{z - x_0} \right)}_{\text{analytic at } x_0}.$$

$f(z) - \dfrac{B}{z - x_0}$ is analytic in $|z - x_0| < R$

Thus $f(z) - \dfrac{B}{z - x_0}$ is bounded in $|z - x_0| \le \dfrac{R}{2}$.

$\exists M > 0$, s.t.

$$\left| f(z) - \frac{B}{z - x_0} \right| \le M \quad \text{for} \quad |z - x_0| \le \frac{R}{2}.$$

$$\left| \int_{C_\rho} \left(f(z) - \frac{B}{z - x_0} \right) dz \right| \le M \cdot \pi \rho \to 0 \quad \text{as } \rho \to 0.$$

$$\int_{C_\rho} \frac{B}{z - x_0}\, dz \xlongequal[dz = \rho i e^{i\theta}]{z = x_0 + \rho e^{i\theta}} \int_\pi^0 \frac{B}{\rho e^{i\theta}} \rho i e^{i\theta}\, d\theta$$

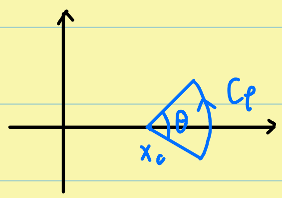
$$= -i \int_0^\pi B\, d\theta = -i \pi B.$$

So $\int_{C_\rho} f(z)\,dz \longrightarrow -i\pi B = -i\pi \operatorname*{Res}_{z=x_0} f(z)$. as $\rho \to 0$ $\square$

RMK. More generally, if f has a simple pole at x_0,

then

$$\int_{C_\rho} f(z)\,dz \longrightarrow i\theta \operatorname*{Res}_{z=x_0} f(z) \quad \text{as } \rho \to 0.$$



The residue of a simple pole x_0 is 'equally distributed' around x_0.