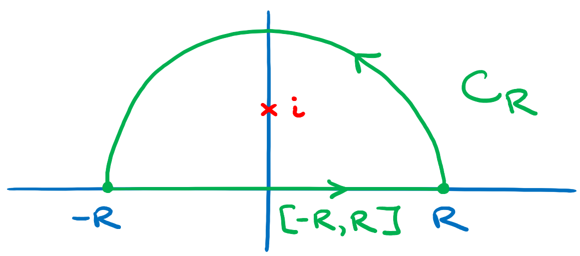
2. Fourier Integrals

We can use residues to evaluate convergent improper integrals of the form

$$\int_{-\infty}^{\infty} f(x)\sin(ax)\,dx, \quad \int_{-\infty}^{\infty} f(x)\cos(ax)\,dx$$

f real valued and $a > 0$.

Example: Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(ax)}{(1+x^2)^2}\,dx$.



Problem: $\dfrac{\cos(az)}{(1+z^2)^2}$ actually grows as $y \to \infty$ since $|\cos(az)|^2 = \cos^2(ax) + \sinh^2(ay)$, and this prevents us from arguing that $\displaystyle\int_{C_R} \frac{\cos(az)}{(1+z^2)^2}\,dz$ goes to zero as $R \to \infty$.

Solution: write $\cos ax = \operatorname{Re} e^{iax}$, compute $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iax}}{(1+x^2)^2}\,dx$ and take the real part at the end.

$$\rightsquigarrow \quad \int_{-\infty}^{\infty} \frac{\cos(ax)}{(1+x^2)^2}\,dx = \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{iax}}{(1+x^2)^2}\,dx$$

C_R : semicircle $z(t) = Re^{it}$, $0 \le t \le \pi$;

Γ_R : closed contour $[-R, R] + C_R$.

$$\int_{\Gamma_R} \frac{e^{iaz}}{(1+z^2)^2}\,dz = 2\pi i \operatorname*{Res}_{z=i} \left(\frac{e^{iaz}}{(1+z^2)^2} \right)$$

Step 1: compute the residue.

Step 2: argue that $\displaystyle\lim_{R\to\infty} \int_{C_R} \frac{e^{iaz}}{(1+z^2)^2}\,dz = 0$.

Step 3: get $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iax}}{(1+x^2)^2}\,dx = \operatorname*{Res}_{z=i} \left(\frac{e^{iaz}}{(1+z^2)^2} \right)$.

Step 4: take real part.

Step 1: $$\frac{e^{iaz}}{(1+z^2)^2} = \frac{e^{iaz}}{(z+i)^2(z-i)^2} = \frac{\phi(z)}{(z-i)^2},$$

$z = i$ is a pole of order 2

$$\phi(z) = \frac{e^{iaz}}{(z+i)^2}, \quad \phi(i) \neq 0.$$

$$\operatorname*{Res}_{z=i} \frac{e^{iaz}}{(1+z^2)^2} = \phi'(i).$$

$$\phi'(z) = \frac{iae^{iaz}}{(z+i)^2} - \frac{2e^{iaz}}{(z+i)^3}$$

$$\rightsquigarrow \quad \phi'(i) = \frac{iae^{-a}}{(2i)^2} - \frac{2e^{-a}}{(2i)^3} = \frac{iae^{-a}}{-4} - \frac{2e^{-a}}{-8i}$$

$$= -\frac{ie^{-a}}{4}(1+a)$$

<u>Step 2:</u> since $a>0$,

$$|e^{iaz}| = |e^{ia(x+iy)}| = |e^{iax}||e^{-ay}| = e^{-ay} \overset{y\geq 0}{\leq} 1$$

for $y \geq 0$.

$$\rightsquigarrow \left| \int_{C_R} \frac{e^{iaz}}{(1+z^2)^2}\, dz \right| \leq \frac{1}{(R^2-1)} \times \pi R \longrightarrow 0 \quad \text{as } R \to \infty.$$

<u>Step 3:</u>

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{(1+x^2)^2}\, dx = 2\pi i \cdot \left(-\frac{ie^{-a}}{4}(1+a) \right)$$

$$= \frac{\pi}{2} e^{-a}(1+a).$$

<u>Step 4:</u>

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{(1+x^2)^2}\, dx = \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{iax}}{(1+x^2)^2}\, dx$$

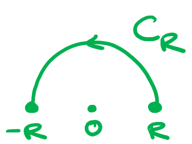
$$= \underline{\underline{\frac{\pi}{2} e^{-a}(1+a).}}$$

<u>Remark:</u> $\frac{\sin(ax)}{(1+x^2)^2}$ is odd, so it is not surprising that $\int_{-\infty}^{\infty} \frac{e^{iax}}{(1+x^2)^2}\, dx$ is real.

Jordan's Lemma:

Example: $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\,dx$.

- Does this integral exist? (Yes!)
- Do we need to take the principal value? (No!)
- Note that our usual method for showing that $\displaystyle\lim_{R\to\infty}\int_{C_R} \frac{z e^{iz}}{z^2+1}\,dz = 0$ doesn't work:



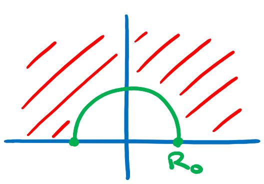
on C_R $\quad \displaystyle\left|\frac{z e^{iz}}{z^2+1}\right| \le \frac{R}{R^2-1}$

$\rightsquigarrow \quad \displaystyle\left|\int_{C_R} \frac{z e^{iz}}{z^2+1}\,dz\right| \le \frac{R}{R^2-1}\cdot \pi R \to \pi$ as $R\to\infty$.

$\rightsquigarrow$ this doesn't tell us whether or not $\displaystyle\int_{C_R} \frac{z e^{iz}}{z^2+1}\,dz \to 0$ as $R\to\infty$.

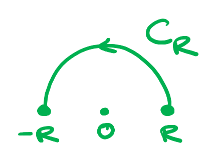
We need Jordan's Lemma.

Theorem (Jordan's Lemma):



$f(z)$ analytic for $\operatorname{Im} z > 0$ and $|z| \ge R_0$;

$C_R: \quad z(t) = Re^{it}, \quad 0 \le t \le \pi;$



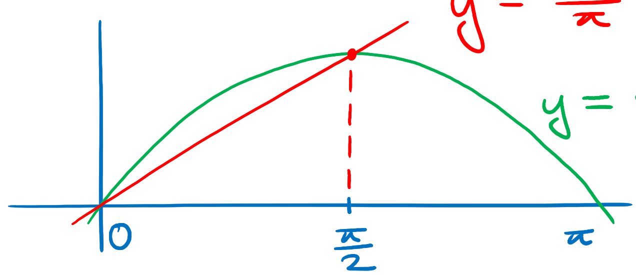
If there is $M_R > 0$ such that $|f(z)| \le M_R$ on C_R (for each $R > R_0$) and $\displaystyle\lim_{R\to\infty} M_R = 0$ then

$$\lim_{R\to\infty}\int_{C_R} f(z)e^{iaz}\,dz = 0.$$

Proof:

We first prove Jordan's Inequality:

$$\int_0^\pi e^{-R\sin\theta}\,d\theta < \frac{\pi}{R}, \qquad R>0.$$



$\sin\theta \geq \frac{2\theta}{\pi}$

for $0\leq\theta\leq\frac{\pi}{2}$

$\rightsquigarrow$ $-R\sin\theta \leq -\frac{2R\theta}{\pi}$ for $0\leq\theta\leq\frac{\pi}{2}$

$\rightsquigarrow$ $$\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\,d\theta \leq \int_0^{\frac{\pi}{2}} e^{-\frac{2R\theta}{\pi}}\,d\theta$$

$$= \frac{\pi}{2R}\left(1-e^{-R}\right) < \frac{\pi}{2R}.$$

$\rightsquigarrow$ By symmetry: $\int_0^\pi e^{-R\sin\theta}\,d\theta < \frac{\pi}{R}$.

Now we can prove the theorem:

$$\left|\int_{C_R} f(z)\,e^{iaz}\,dz\right| = \left|\int_0^\pi f(Re^{i\theta})\,e^{iaRe^{i\theta}}\cdot iRe^{i\theta}\,d\theta\right|$$

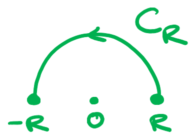
$$\leq \int_0^\pi M_R\left|e^{iaRe^{i\theta}}\right|\cdot R\,d\theta \xrightarrow{e^{i\theta}=\cos\theta+i\sin\theta} = M_R\cdot R\int_0^\pi e^{-aR\sin\theta}\,d\theta$$

$$\leq M_R\cdot R\cdot\frac{\pi}{aR} = \frac{\pi M_R}{a} \longrightarrow 0$$

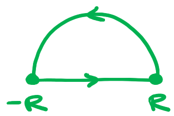
as $R\to\infty$. $\square$

Back to our example: $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\,dx$.

$$f(z) = \frac{z}{z^2+1}$$



$C_R : \ z(t) = Re^{it}, \quad 0 \le t \le \pi.$



$\Gamma_R : \ [-R, R] + C_R, \quad R > 1.$

$\rightsquigarrow$ On C_R $\quad |f(z)| \le \underbrace{\frac{R}{R^2-1}}_{\text{"}M_R\text{"}} \to 0$ as $R \to \infty$.

$\rightsquigarrow$ We can apply Jordan's Lemma.

$$\rightsquigarrow \lim_{R\to\infty} \int_{C_R} f(z)\, e^{iz}\, dz = 0.$$

Applying the Residue Thm to Γ_R and letting $R \to \infty$,

$$\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+1}\,dx = 2\pi i \cdot \operatorname*{Res}_{z=i} \frac{z e^{iz}}{z^2+1}.$$

Compute residue:

$$\operatorname*{Res}_{z=i} \frac{z e^{iz}}{z^2+1} = \left.\frac{z e^{iz}}{z+i}\right|_{z=i} = \frac{i e^{i^2}}{2i} = \frac{1}{2e}.$$

$$\rightsquigarrow \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+1}\,dx = \frac{\pi i}{e}.$$

take real and imaginary part

So $\displaystyle\int_{-\infty}^{\infty} \underbrace{\frac{x \cos x}{x^2+1}}_{\text{odd function}}\,dx = 0$, $\quad \boxed{\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\,dx = \frac{\pi}{e}.}$