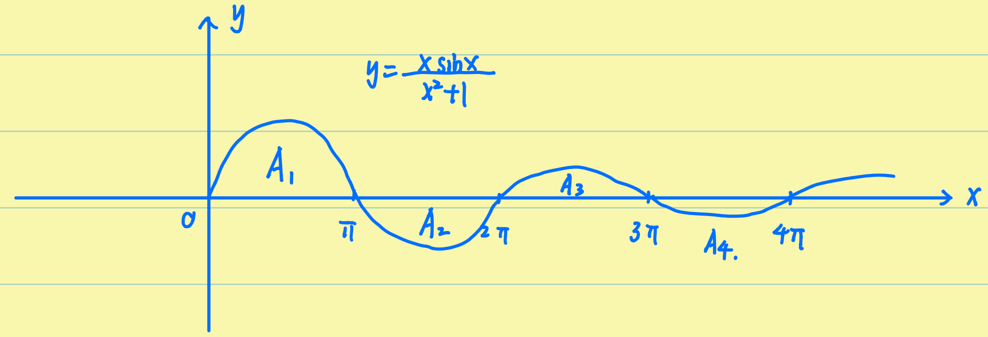
§ 88. Jordan's lemma.

Exercise. $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\, dx$.

- Is the integral convergent?

$\left|\frac{x}{x^2+1}\right| \longrightarrow 0$ as $x \to \pm\infty$.



$\int_0^{\infty} \frac{x \sin x}{x^2+1}\, dx = \underbrace{A_1 - A_2 + A_3 - A_4 \cdots}_{\text{alternating series and } A_j \downarrow 0 \text{ as } j \to \infty.}$

By the alternating series test from Calculus,

$\int_0^{\infty} \frac{x \sin x}{x^2+1}\, dx$ is convergent.

Similarly, $\int_{-\infty}^{0} \frac{x \sin x}{x^2+1}\, dx$ is also convergent.

So $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\, dx$ is convergent.

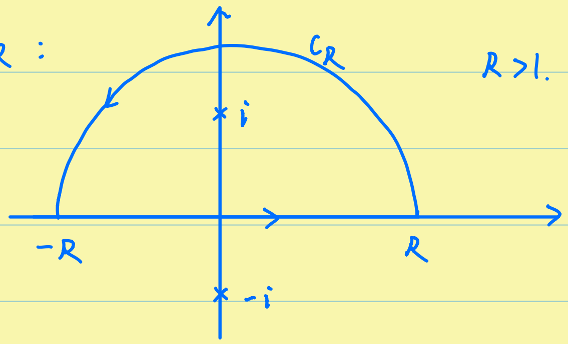
- Compute $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\, dx$.

Set $f(z) = \frac{z e^{iz}}{z^2+1}$.

Then $f(z) = \frac{z \cos z}{z^2+1} + i \underbrace{\frac{z \sin z}{z^2+1}}_{\text{the imaginary part is the integrand.}}$.

Singularities : $z^2+1=0$ $\quad z=\pm i$.

Γ_R :



$R>1$.

$$\int_{\Gamma_R} f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx.$$

Problem. Our estimate to show $\int_{C_R} f(z)\,dz \to 0$ as $R\to\infty$ does not work.

For $z \in C_R$,

$$|f(z)| = \left|\frac{z e^{iz}}{z^2+1}\right| = \frac{|z|\cdot e^{-y}}{|z^2+1|} \le \frac{R}{R^2-1}.$$

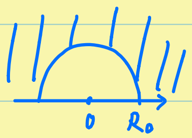
$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{R}{R^2-1}\cdot \pi R \longrightarrow \pi.$$

as $R\to\infty$.

This does not tell us whether or not

$\int_{C_R} f(z)\,dz \to 0$ as $R\to\infty$.

Solution: We need a more precise estimate.



Thm. (Jordan's Lemma).

- $f(z)$ is analytic for $\operatorname{Im} z > 0$ and $|z| \geq R_0$.
- C_R : $z(t) = Re^{it}$ $\quad t \in [0, \pi]$.
- $\exists\, M_R > 0$ s.t.

$$|f(z)| \leq M_R \quad \text{for any } z \in C_R \quad (R > R_0)$$

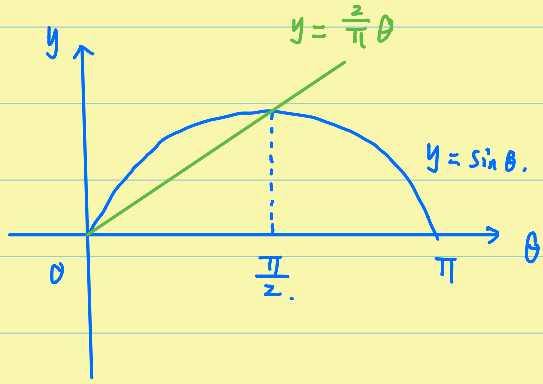
and $\lim_{R\to\infty} M_R = 0$.

Then for any $a > 0$,

$$\lim_{R\to\infty} \int_{C_R} f(z)\, e^{iaz}\, dz = 0.$$

Proof. The key ingredient : Jordan's inequality.

$$\int_0^{\pi} e^{-R\sin\theta}\, d\theta < \frac{\pi}{R} \quad \text{for any } R > 0.$$



$\sin\theta \geq \frac{2}{\pi}\theta$

for $\theta \in [0, \frac{\pi}{2}]$.

$$e^{-R\sin\theta} \leq e^{-R\cdot\frac{2}{\pi}\theta} \quad \text{for } \theta \in [0, \tfrac{\pi}{2}].$$

$$\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\, d\theta \le \int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi}\theta}\, d\theta$$
$$= -\frac{\pi}{2R} e^{-\frac{2R}{\pi}\theta}\Big|_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{2R}\left(1 - e^{-R}\right)$$
$$< \frac{\pi}{2R}.$$

By symmetry, $\int_0^{\pi} e^{-R\sin\theta}\, d\theta = 2\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\, d\theta < \frac{\pi}{R}$.

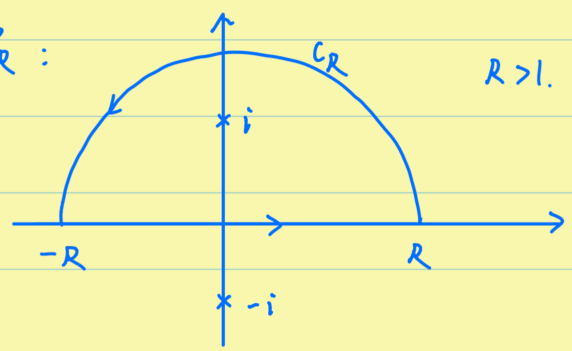
This verifies Jordan's inequality.

Now we can prove the theorem.

$$\left|\int_{C_R} f(z)\, e^{iaz}\, dz\right| \overset{z=Re^{i\theta}}{=\!=} \left|\int_0^{\pi} f(Re^{i\theta})\, e^{iaRe^{i\theta}}\, R\cdot i e^{i\theta}\, d\theta\right|$$
$$\le \int_0^{\pi} \left| f(Re^{i\theta})\, e^{iaRe^{i\theta}}\, R\, i e^{i\theta}\right| d\theta$$
$$= R\int_0^{\pi} \left|f(Re^{i\theta})\right| e^{-aR\sin\theta}\, d\theta$$
$$\le R\, M_R \int_0^{\pi} e^{-aR\sin\theta}\, d\theta$$
$$\xrightarrow{\text{Jordan's inequality}} \; < R\, M_R\, \frac{\pi}{aR}$$
$$= \frac{\pi}{a} M_R \xrightarrow[\uparrow\; M_R \to 0.]{} 0 \quad \text{as } R \to \infty. \quad \square$$

Back to our exercise. $\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+1}\, dx$.

Γ_R :



$R > 1$.

$$f(z) = \frac{z\, e^{iz}}{z^2+1}$$

$$\int_{P_R} f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-\infty}^{\infty} f(x)\,dx.$$

① ②

① $\int_{P_R} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z)$

$$f(z) = \frac{1}{z-i}\cdot\frac{z e^{iz}}{z+i} \triangleq \frac{\phi(z)}{z-i}.$$

$$\operatorname*{Res}_{z=i} f(z) = \phi(i) = \frac{i e^{-1}}{2i} = \frac{1}{2}e^{-1}.$$

$$\int_{P_R} f(z)\,dz = \pi i e^{-1}.$$

② $\int_{C_R} f(z)\,dz = \int_{C_R} \frac{z}{z^2+1} e^{iz}\,dz.$

For $z \in C_R$,

$$\left|\frac{z}{z^2+1}\right| \le \underbrace{\frac{R}{R^2-1}}_{M_R} \to 0 \quad \text{as } R\to\infty.$$

By Jordan's lemma,

$$\int_{C_R} f(z)\,dz \to 0 \quad \text{as } R\to\infty.$$

So $\int_{-\infty}^{\infty} \frac{x}{x^2+1} e^{ix}\,dx = i\frac{\pi}{e}.$

Take the imaginary parts: $\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+1}\,dx = \frac{\pi}{e}.$

Take the real parts: $\int_{-\infty}^{\infty} \underbrace{\frac{x\cos x}{x^2+1}}_{\text{odd function}}\,dx = 0.$