20. Applications of Residues: Improper Integrals

① Improper integrals & Cauchy P.V.:

Def^n: $f(x)$ continuous real function,

$$\int_{-\infty}^{\infty} f(x)\,dx \overset{\text{def}^n}{=} \lim_{R_1\to\infty}\int_{-R_1}^{0} f(x)\,dx + \lim_{R_2\to\infty}\int_{0}^{R_2} f(x)\,dx$$

provided these limits exist.

Otherwise we say $\int_{-\infty}^{\infty} f(x)\,dx$ diverges.

Cauchy principal value:

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx.$$

- The principal value integral may exist when $\int_{-\infty}^{\infty} f(x)\,dx$ does not.
- If $\int_{-\infty}^{\infty} f(x)\,dx$ exists, then

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

- If $f(x)$ is even $(f(x) = f(-x))$ then the P.V. integral exists if and only if the full improper integral exists and

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 2\int_{0}^{\infty} f(x)\,dx.$$

<u>Example</u>:

$$\text{P.V.} \int_{-\infty}^{\infty} x\,dx = \lim_{R\to\infty} \int_{-R}^{R} x\,dx$$

$$= \lim_{R\to\infty} \left[\frac{x^2}{2}\right]_{-R}^{R} = 0.$$

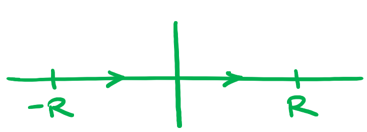
<u>BUT</u> $\int_{-\infty}^{\infty} x\,dx$ does not exist!

② <u>Improper integrals of rational functions</u>

Consider rational functions $\frac{p(x)}{q(x)}$ where $p(x)$ & $q(x)$ are real polynomials with $q(x)$ having no real zeros.

<u>Examples</u>: $\frac{x^2+2x+1}{x^2+1}$, $\frac{1}{x^6+1}$.

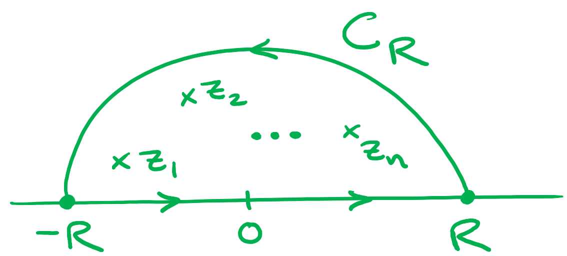
$$\text{P.V.} \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{p(x)}{q(x)}\,dx$$



$$= \lim_{R\to\infty} \int_{[-R,R]} \frac{p(z)}{q(z)}\,dz.$$

Strategy to compute P.V. $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx$.

<u>Step 1</u>: For $R>0$ let C_R denote the semicircle $|z| = R$, $\operatorname{Im} z \geq 0$, oriented positively and let Γ_R denote the simple closed contour $[-R,R] + C_R$.



Take R large enough so that all of the poles of $\frac{p(z)}{q(z)}$ lying in the upper half plane are interior to Γ_R.

<u>Step 2</u>: Apply the Residue Theorem to $\int_{\Gamma_R} \frac{p(z)}{q(z)}\,dz$ to get

$$\int_{[-R,R]} \frac{p(z)}{q(z)}\,dz + \int_{C_R} \frac{p(z)}{q(z)}\,dz = 2\pi i \sum_{j=1}^{n} \operatorname*{Res}_{z=z_j} \frac{p(z)}{q(z)}.$$

<u>Step 3</u>: Show that $\lim_{R\to\infty} \int_{C_R} \frac{p(z)}{q(z)}\,dz = 0$.

Here we need $\frac{p(z)}{q(z)}$ to decay fast enough as $z \to \infty$.

<u>Step 4</u>: $\text{P.V.} \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx = 2\pi i \sum_{j=0}^{n} \underbrace{\operatorname*{Res}_{z=z_j} \frac{p(z)}{q(z)}}_{\text{compute!}}$.

Examples:

① Use residues to evaluate the integral

$$\int_0^\infty \frac{dx}{x^2+4}.$$

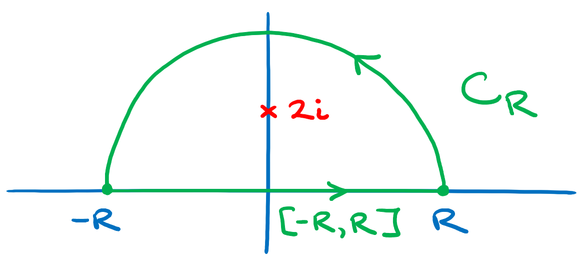
Recall $\displaystyle\int_0^\infty \frac{dx}{x^2+4} = \lim_{R\to\infty} \int_0^R \frac{dx}{x^2+4}.$

Since $\dfrac{1}{x^2+4}$ is even, $\displaystyle\int_0^R \frac{dx}{x^2+4} = \frac{1}{2}\int_{-R}^R \frac{dx}{x^2+4}.$

Consider $f(z) = \dfrac{1}{z^2+4}$, and define

C_R : semicircle $z(t) = Re^{it}$, $0 \le t \le \pi$;

Γ_R : simple closed contour $\underbrace{[-R,R]}_{\text{straight line from } -R \text{ to } R} + C_R$.



$f(z) = \dfrac{1}{z^2+4} = \dfrac{1}{(z+2i)(z-2i)}$ has simple poles at $z = 2i$ and $z = -2i$.

$\rightsquigarrow \operatorname*{Res}_{z=2i} f(z) = \left.\dfrac{1}{z+2i}\right|_{z=2i} = \dfrac{1}{4i}$

$f(z) = \dfrac{\phi(z)}{z-2i}$, $\phi(z) = \dfrac{1}{z+2i}$

$\rightsquigarrow \operatorname*{Res}_{z=2i} f(z) = \phi(2i) = \dfrac{1}{4i}$

$\rightsquigarrow \displaystyle\int_{\Gamma_R} f(z)\,dz = 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2}.$

$$\rightsquigarrow \quad \underbrace{\int_{\Gamma_R} f(z)\,dz}_{=\frac{\pi}{2}} = \underbrace{\int_{-R}^{R} f(x)\,dx}_{\int_{[-R,R]} f(z)\,dz} + \int_{C_R} f(z)\,dz$$

Now let $R \to \infty$.

Claim: $\displaystyle\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0.$

Proof of claim: For $z \in C_R$ $(R > 2)$

$$|f(z)| = \frac{1}{|z^2+4|} \leq \frac{1}{||z|^2-4|} = \frac{1}{R^2-4}$$

Reverse triangle inequality

$$\text{Length}(C_R) = \pi R$$

$$\rightsquigarrow \left| \int_{C_R} f(z)\,dz \right| \leq \frac{1}{R^2-4} \cdot \pi R = \frac{\pi R}{R^2-4},$$

$$\frac{\pi R}{R^2-4} \to 0 \text{ as } R \to \infty.$$

$$\rightsquigarrow \quad \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = \frac{\pi}{2}.$$

$$\rightsquigarrow \quad \int_0^\infty \frac{dx}{x^2+4} = \lim_{R\to\infty} \int_0^R f(x)\,dx$$

$$= \frac{1}{2} \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = \underline{\frac{\pi}{4}}.$$

② Compute $\displaystyle\int_0^\infty \frac{dx}{x^6+1}$.

$f(z) = \dfrac{1}{z^6+1}$ has simple poles at the 6^{th} roots of -1.

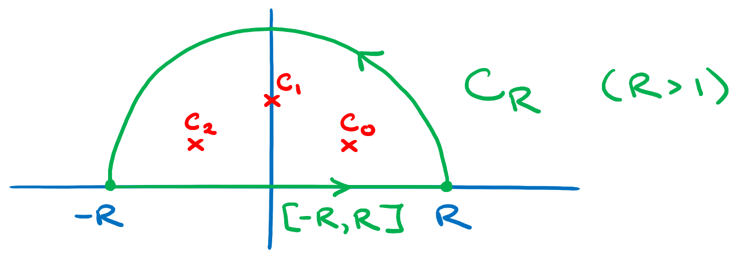
The 6^{th} roots of $-1 = e^{i\pi}$ are:

$$c_k = \exp\left[i\left(\frac{\pi}{6} + \frac{2k\pi}{6}\right)\right] \qquad k = 0,1,2,3,4,5.$$

$$z^6 + 1 = (z-c_0)(z-c_1)(z-c_2)(z-c_3)(z-c_4)(z-c_5)$$

Define C_R : semicircle $z(t) = Re^{it}$, $0 \le t \le \pi$;

Γ_R : closed contour $[-R,R] + C_R$.



Write $f(z) = \dfrac{1}{z^6+1} = \dfrac{p(z)}{q(z)}$ where

$p(z) = 1$ & $q(z) = z^6 + 1$. Since each

c_k is a simple pole,

$$\operatorname*{Res}_{z=c_k} f(z) = \frac{p(c_k)}{q'(c_k)} = \frac{1}{6c_k^5} = \frac{c_k}{6c_k^6} = -\frac{c_k}{6}$$

for each k.

Residue Thm.

$$\rightsquigarrow \int_{\Gamma_R} f(z)\,dz = 2\pi i \cdot \left(\frac{-1}{6}\right)(c_0 + c_1 + c_2)$$

$$= 2\pi i \cdot \left(\frac{-1}{6}\right)\Big(\underbrace{e^{i\pi/6}}_{\frac{\sqrt{3}}{2}+\frac{i}{2}} + i - \underbrace{e^{-i\pi/6}}_{-\frac{\sqrt{3}}{2}+\frac{i}{2}}\Big)$$

$$= 2\pi i \cdot \left(\frac{-1}{6}\right)(2i) = \frac{2\pi}{3}.$$

$$\rightsquigarrow \frac{2\pi}{3} = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz$$

Again we argue that $\displaystyle\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0.$

For $z \in C_R$, $\quad |z^6+1| \geq \big|\,|z|^6 - 1\big| = R^6 - 1 \quad (R>1)$

$\rightsquigarrow \quad |f(z)| \leq \dfrac{1}{R^6-1}$

$\rightsquigarrow \quad \left|\displaystyle\int_{C_R} f(z)\,dz\right| \leq \dfrac{\pi R}{R^6-1} \longrightarrow 0$ as $R\to\infty$

$$\rightsquigarrow \quad \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = \frac{2\pi}{3}$$

$$\rightsquigarrow \quad \int_0^\infty \frac{dx}{x^6+1} = \lim_{R\to\infty}\int_0^R f(x)\,dx$$

$$= \frac{1}{2}\cdot\frac{2\pi}{3} = \underline{\underline{\frac{\pi}{3}}}.$$

(Topic 3)

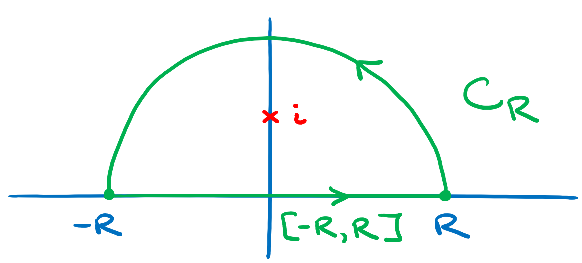
(3) Fourier Integrals

We can use residues to evaluate convergent improper integrals of the form

$$\int_{-\infty}^{\infty} f(x)\sin(ax)\,dx, \qquad \int_{-\infty}^{\infty} f(x)\cos(ax)\,dx$$

f real valued and $a>0$.

Example: Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(ax)}{(1+x^2)^2}\,dx.$



Problem: $\frac{\cos(az)}{(1+z^2)^2}$ actually grows as $y \to \infty$ since $|\cos(az)|^2 = \cos^2(ax) + \underline{\sinh^2(ay)}$, and this prevents us from arguing that $\int_{C_R} \frac{\cos(az)}{(1+z^2)^2}\,dz$ goes to zero as $R \to \infty$.

Solution: write $\cos ax = \operatorname{Re} e^{iax}$, compute $\int_{-\infty}^{\infty} \frac{e^{iax}}{(1+x^2)^2}\,dx$ and take the real part at the end.

In general we want P.V. $\int_{-\infty}^{\infty}$ but here we are okay.

$$\rightsquigarrow \int_{-\infty}^{\infty} \frac{\cos(ax)}{(1+x^2)^2}\,dx = \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{iax}}{(1+x^2)^2}\,dx$$

C_R : semicircle $z(t) = Re^{it}$, $0 \le t \le \pi$;

Γ_R : closed contour $[-R, R] + C_R$.

$$\int_{\Gamma_R} \frac{e^{iaz}}{(1+z^2)^2}\,dz = 2\pi i \operatorname*{Res}_{z=i} \left(\frac{e^{iaz}}{(1+z^2)^2} \right)$$

Step 1: compute the residue.

Step 2: argue that $\lim_{R\to\infty} \int_{C_R} \frac{e^{iaz}}{(1+z^2)^2}\,dz = 0$.

Step 3: get $\int_{-\infty}^{\infty} \frac{e^{iax}}{(1+x^2)^2}\,dx = \operatorname*{Res}_{z=i} \left(\frac{e^{iaz}}{(1+z^2)^2} \right)$.

Step 4: take real part.

Step 1: $\dfrac{e^{iaz}}{(1+z^2)^2} = \dfrac{e^{iaz}}{(z+i)^2(z-i)^2} = \dfrac{\phi(z)}{(z-i)^2}$,

$z=i$ is a pole of order 2

$\phi(z) = \dfrac{e^{iaz}}{(z+i)^2}$, $\phi(i) \neq 0$.

$$\operatorname*{Res}_{z=i} \frac{e^{iaz}}{(1+z^2)^2} = \phi'(i).$$

$$\phi'(z) = \frac{iae^{iaz}}{(z+i)^2} - \frac{2e^{iaz}}{(z+i)^3}$$

$$\rightsquigarrow \phi'(i) = \frac{iae^{-a}}{(2i)^2} - \frac{2e^{-a}}{(2i)^3} = \frac{iae^{-a}}{-4} - \frac{2e^{-a}}{8i}$$

$$= -\frac{ie^{-a}}{4}(1+a)$$

Step 2: since $a>0$,

$$|e^{iaz}| = |e^{ia(x+iy)}| = |e^{iax}||e^{-ay}| = e^{-ay} \leq 1$$

$y \geq 0$

for $y \geq 0$.

$$\rightsquigarrow \left| \int_{C_R} \frac{e^{iaz}}{(1+z^2)^2}\, dz \right| \leq \frac{1}{(R^2-1)} \times \pi R \longrightarrow 0$$

as $R \to \infty$.

Step 3: $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iax}}{(1+x^2)^2}\, dx = 2\pi i \cdot \left(-\frac{ie^{-a}}{4}(1+a)\right)$

$$= \frac{\pi}{2} e^{-a}(1+a).$$

Step 4: $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(ax)}{(1+x^2)^2}\, dx = \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{iax}}{(1+x^2)^2}\, dx$

$$= \frac{\pi}{2} e^{-a}(1+a).$$

Remark: $\frac{\sin(ax)}{(1+x^2)^2}$ is odd, so it is not surprising that $\int_{-\infty}^{\infty} \frac{e^{iax}}{(1+x^2)^2}\, dx$ is real.