§ 87. Improper integrals from Fourier analysis. 1.

We can use residues to evaluate convergent improper integrals of the form

$$\int_{-\infty}^{\infty} f(x)\ \sin(ax)\ dx \ , \qquad \int_{-\infty}^{\infty} f(x)\ \cos(ax)\ dx.$$

f is real valued and $a > 0$.

E.g. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(ax)}{(1+x^2)^2}\, dx$. $a>0$.

Is it convergent ?

$$\left|\frac{\cos(ax)}{(1+x^2)^2}\right| \le \frac{1}{(1+x^2)^2} \sim \frac{1}{x^4} \quad \text{as} \quad x \to \pm\infty.$$

So it is convergent by the P-test.

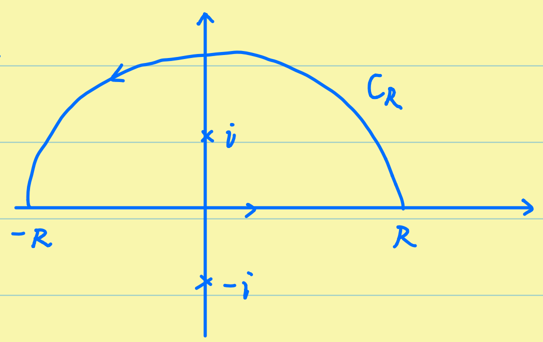
Set $\displaystyle f(z) = \frac{\cos(az)}{(1+z^2)^2}$.

$(1+z^2)^2 = 0$ $\qquad 1+z^2 = 0$

$z^2 = -1$

$z = i \ , \ -i$.

Γ_R



$$\int_{\Gamma_R} f(z)\, dz = \int_{C_R} f(z)\, dz + \int_{-R}^{R} f(x)\, dx.$$

Sec. 39
example 2.

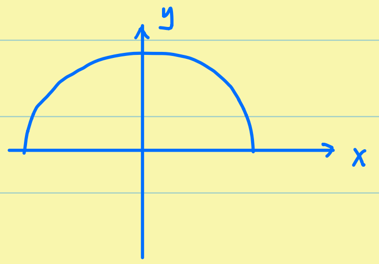
Problem: For $z \overset{\text{in}}{\in} C_R$, $|f(z)| \not\to 0$ as $R \to \infty$.

Recall

$$|\cos(az)|^2 = \underbrace{\cos^2(ax)}_{\text{bounded}} + \underbrace{\sinh^2(ay)}_{\left(\frac{e^{ay}-e^{-ay}}{2}\right)^2 \sim \left(\frac{e^{ay}}{2}\right)^2 \text{ as } y\to\infty}.$$

$$|f(z)| = \frac{|\cos(az)|}{|(1+z^2)^2|} \underset{=}{\overset{z=iy}{=}} \frac{|\sinh(ay)|}{(1+y^2)^2} \sim \frac{1}{2}\frac{e^{ay}}{y^4} \to \infty$$

as $y \to \infty$.



$|f(z)| \to \infty$ along the y-axis as $y \to \infty$.

This prevents us from arguing $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$.

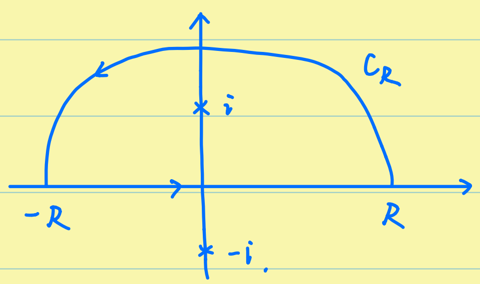
Solution: $e^{iax} = \cos(ax) + i\sin(ax)$.

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{(1+x^2)^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos(ax)}{(1+x^2)^2}\,dx + i\int_{-\infty}^{\infty} \frac{\sin(ax)}{(1+x^2)^2}\,dx.$$

We compute $\int_{-\infty}^{\infty} \frac{e^{iax}}{(1+x^2)^2}\,dx$ and take its real part.

Set $h(z) = \dfrac{e^{iaz}}{(1+z^2)^2}$.

Γ_R



(*) $\int_{\Gamma_R} h(z)\,dz = \int_{C_R} h(z)\,dz + \int_{-R}^{R} h(x)\,dx.$

① ②.

①. $\int_{\Gamma_R} h(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} h(z)$.

$$h(z) = \frac{e^{iaz}}{(1+z^2)^2} = \frac{1}{(z-i)^2} \cdot \frac{e^{iaz}}{(z+i)^2} \triangleq \frac{\phi(z)}{(z-i)^2}.$$

$\phi(z) = \frac{e^{iaz}}{(z+i)^2}$. analytic at i.

$\phi(i) = \frac{e^{-a}}{(2i)^2} = -\frac{e^{-a}}{4}.$

$\operatorname*{Res}_{z=i} h(z) = \phi'(i).$

$$\phi'(z) = \frac{ia\,e^{iaz}(z+i)^2 - e^{iaz}\cdot 2(z+i)}{(z+i)^4} = \frac{ia(z+i) - 2}{(z+i)^3}\,e^{iaz}$$

$$\phi'(i) = \frac{ia\cdot 2i - 2}{(2i)^3}\,e^{-a} = \frac{-2a-2}{-8i}\,e^{-a} = \frac{a+1}{4i}\,e^{-a}.$$

$$\int_{\Gamma_R} h(z)\,dz = 2\pi i \cdot \frac{a+1}{4i}\,e^{-a} = \frac{\pi(a+1)}{2}\,e^{-a}.$$

②. $\int_{C_R} h(z)\,dz$. $h(z) = \frac{e^{iaz}}{(1+z^2)^2}$.

For $z \in C_R$,

$$|e^{iaz}| = |e^{ia(x+iy)}| = e^{-ay} \underset{a>0\ y>0}{\le} 1.$$

$$|h(z)| \le \frac{1}{|(1+z^2)^2|} \le \frac{1}{(|z|^2-1)^2} = \frac{1}{(R^2-1)^2}.$$

$$\left| \int_{C_R} h(z)\, dz \right| \le \frac{1}{(R^2-1)^2} \cdot \pi R \longrightarrow 0 \quad \text{as} \quad R \to \infty.$$

Let $R \to \infty$ in $(*)$.

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{(1+x^2)^2}\, dx = \frac{\pi}{2}(a+1)e^{-a}.$$

Take the real parts on both sides.

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{(1+x^2)^2}\, dx = \frac{\pi}{2}(a+1)\, e^{-a}.$$

<u>RMK.</u> If we take imaginary parts instead,

then

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{(1+x^2)^2}\, dx = 0.$$

This is not surprising since $\frac{\sin(ax)}{(1+x^2)^2}$ is odd.