§ 85 and 86. Evaluation of improper integrals.

Def. $f(x)$ is a continuous real function.

$$\int_0^\infty f(x)\,dx \overset{\text{def.}}{=} \lim_{R\to+\infty} \int_0^R f(x)\,dx,$$

defined by

$$\int_{-\infty}^0 f(x)\,dx \overset{\text{def.}}{=} \lim_{R\to+\infty} \int_{-R}^0 f(x)\,dx = \lim_{R\to-\infty} \int_R^0 f(x)\,dx,$$

$$\int_{-\infty}^\infty f(x)\,dx \overset{\text{def.}}{=} \int_0^\infty f(x)\,dx + \int_{-\infty}^0 f(x)\,dx$$

$$= \underline{\lim_{R_1\to+\infty} \int_0^{R_1} f(x)\,dx} + \underline{\lim_{R_2\to+\infty} \int_{-R_2}^0 f(x)\,dx},$$

consider ∞ and $-\infty$ separately.

$$\text{P.V.} \int_{-\infty}^{+\infty} f(x)\,dx \overset{\text{def.}}{=} \underline{\lim_{R\to+\infty} \int_{-R}^R f(x)\,dx},$$

Cauchy principal value. take the limit in a symmetric way.

provided these limits exist.

E.g. $\int_{-\infty}^\infty x\,dx$. and $\text{P.V.} \int_{-\infty}^\infty x\,dx$.

$$\int_{-\infty}^\infty x\,dx = \int_0^\infty x\,dx + \int_{-\infty}^0 x\,dx.$$

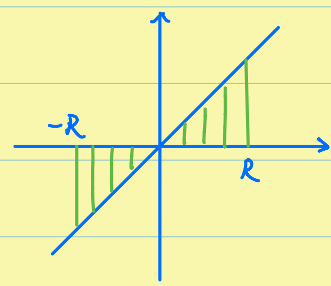
$$\int_0^\infty x\,dx = \lim_{R\to+\infty} \int_0^R x\,dx = \lim_{R\to+\infty} \frac{1}{2}R^2 = \infty. \quad \text{D.N.E.}$$

So $\int_{-\infty}^\infty x\,dx$ does not exist.

$$\text{P.V.} \int_{-\infty}^\infty x\,dx = \lim_{R\to+\infty} \int_{-R}^R x\,dx$$

$$= \lim_{R\to+\infty} \left(\frac{1}{2}R^2 - \frac{1}{2}(-R)^2\right)$$

$$= 0.$$



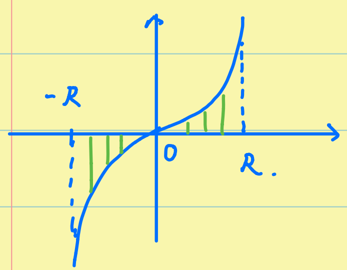
RMK. If $\int_{-\infty}^{\infty} f(x)\,dx$ exists,

then P.V. $\int_{-\infty}^{\infty} f(x)\,dx$ exists and

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

RMK. If $f(x)$ is an odd function, i.e., $f(-x) = -f(x)$,

then

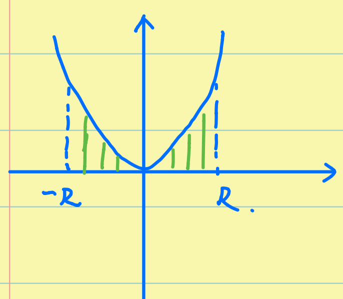


$$\int_{-R}^{R} f(x) = 0,$$

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = 0.$$

RMK. If $f(x)$ is an even function, i.e., $f(-x) = f(x)$,

then



$$\int_{-R}^{R} f(x) = 2\int_{0}^{R} f(x)\,dx = 2\int_{-R}^{0} f(x)\,dx$$

$$\text{P.V.} \int_{-\infty}^{\infty} f(x) = 2\int_{0}^{\infty} f(x)\,dx = 2\int_{-\infty}^{0} f(x)\,dx.$$

provided the limits exist.

E.g. Compute $\int_{0}^{\infty} \frac{dx}{(x^2+1)^2}$.

First, is this integral convergent?

$f(x) = \frac{1}{(x^2+1)^2} \sim \frac{1}{x^4}$ as $x \to \infty$.

Recall the P-test from Calculus (Math 20B)

$$\int_1^\infty \frac{1}{x^p}\,dx \begin{cases} \text{converges} & \text{if } p>1 \\ \text{diverges} & \text{if } p\le 1. \end{cases}$$

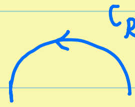
So our integral $\int_0^\infty \frac{1}{(1+x^2)^2}\,dx$ converges.

Next, let's evaluated it.

the integrand function is even.

$$\int_0^\infty \frac{1}{(1+x^2)^2}\,dx = \lim_{R\to\infty} \int_0^R \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{1}{(1+x^2)^2}\,dx.$$

C_R

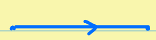


Define $\underline{C_R}$: $z(t) = Re^{it}$ $t \in [0,\pi]$.

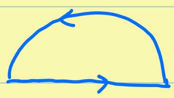
in

simi circle Contour

$[-R, R]$.



Contour $[-R, R]$: $z(t) = t$ $t\in[-R,R]$.



Γ_R.

Simple closed contour Γ_R : $C_R + [-R, R]$.

Set $f(z) = \frac{1}{(1+z^2)^2}$. extending $\frac{1}{(1+x^2)^2}$ to the complex plane.

$$\int_{\Gamma_R} f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{[-R,R]} f(z)\,dz.$$

$\uparrow$ Compute it by residues. in Γ_R.

$\uparrow$ going to 0 as $R\to\infty$.

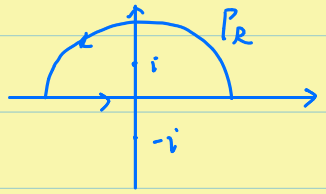
$\| \; z=x, \; dz = dx.$

$\int_{-R}^{R} f(x)\,dx.$

① ② ③.

① $f(z) = \dfrac{1}{(1+z^2)^2}$

$(z^2+1)^2 = 0 \Rightarrow z^2+1 = 0 \Rightarrow z = \pm i.$



For $R > 1$,

$\int_{\Gamma_R} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z)$

denoted by

$f(z) = \dfrac{1}{(z+i)^2} \cdot \dfrac{1}{(z-i)^2} \triangleq \dfrac{\phi(z)}{(z-i)^2}.$

$\phi(z) = \dfrac{1}{(z+i)^2}$ is analytic and nonzero at i.

$z = i$ is a pole of order 2.

$\operatorname*{Res}_{i} f(z) = \phi'(z)\Big|_{z=i} = \dfrac{-2}{(z+i)^3}\Big|_{z=i} = \dfrac{-2}{-8i} = -\dfrac{i}{4}.$

$\int_{\Gamma_R} f(z)\,dz = \dfrac{\pi}{2}.$

② $\int_{C_R} f(z)\,dz.$

Claim: $\lim\limits_{R\to\infty} \int_{C_R} f(z)\,dz = 0.$

For $z \in C_R$, $|f(z)| = \dfrac{1}{|1+z^2|^2}.$

$|1+z^2| \geq |z|^2 - 1 = R^2 - 1.$

↑ triangle inequality.

$|f(z)| \leq \dfrac{1}{(R^2-1)^2}$

Sec 47. $\left|\int_{C_R} f(z)\,dz\right| \leq \dfrac{1}{(R^2-1)^2} \cdot \text{length of } C_R = \dfrac{\pi R}{(R^2-1)^2} \to 0$ as $R \to \infty$.

③ $\frac{1}{2}\int_{[-R,R]} f(z)\,dz \;\underset{dz=dx}{\overset{z=x}{=\!=}}\; \frac{1}{2}\int_{-R}^{R} f(x)\,dx.$

$$\int_{P_R} f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{[-R,R]} f(z)\,dz.$$

Letting $R \to \infty$,

$$\frac{\pi}{2} = 0 + \int_{-\infty}^{\infty} f(x)\,dx.$$

So $\displaystyle\int_0^\infty \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{4}.$

E.g. Compute $\displaystyle\int_0^\infty \frac{dx}{x^4+1}$.

Is this integral convergent?

$f(x) = \frac{1}{x^4+1} \sim \frac{1}{x^4}$ as $x \to \infty$.

$\int_0^\infty f(x)\,dx$ is convergent by the P-test.

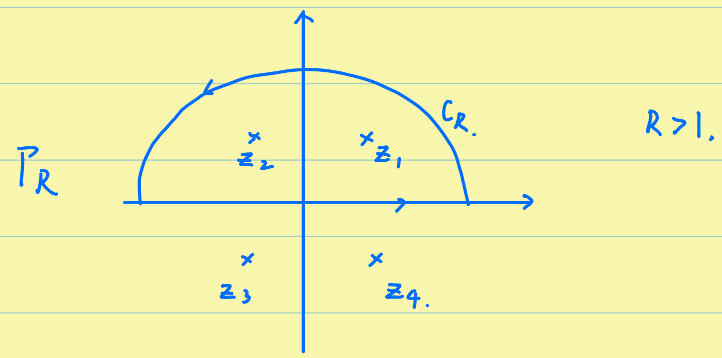
Set $f(z) = \frac{1}{z^4+1}$.

$z^4+1 = 0$

$z^4 = -1 = e^{i\pi}$

$z = e^{\frac{1}{4}(i\pi + i2k\pi)}$ $\quad k \in \mathbb{Z}$.

$z = \underset{z_1}{e^{i\cdot\frac{\pi}{4}}},\ \underset{z_2}{e^{i\cdot\frac{3\pi}{4}}},\ \underset{z_3}{e^{i\frac{5\pi}{4}}},\ \underset{z_4}{e^{i\frac{7\pi}{4}}}.$



(*) $\int_{\Gamma_R} f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx.$

Compute residue: $\int_{\Gamma_R} f(z)\,dz = 2\pi i \left(\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) \right)$

$$f(z) = \frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}.$$

z_1, z_2, z_3, z_4 are simple poles.

$z_1^4 = -1$

$$\operatorname*{Res}_{z=z_1} f(z) = \left.\frac{1}{(z^4+1)'}\right|_{z=z_1} = \frac{1}{4z_1^3} = -\frac{z_1}{4} = -\frac{1}{4}e^{i\frac{\pi}{4}}.$$

$$\operatorname*{Res}_{z=z_2} f(z) = \frac{1}{4z_2^3} = -\frac{z_2}{4} = -\frac{1}{4}e^{i\frac{3\pi}{4}}.$$

$$\int_{\Gamma_R} f(z)\,dz = -2\pi i \cdot \frac{1}{4}\left(e^{i\frac{\pi}{4}} + e^{i\frac{3}{4}\pi} \right)$$
$$= -\frac{\pi i}{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \right)$$
$$= \frac{\sqrt{2}}{2}\pi.$$

Estimate: For $z \in C_R$, $|z^4+1| \geq |z|^4 - 1 = R^4 - 1$,

$$|f(z)| \leq \frac{1}{R^4-1}.$$

$$\left| \int_{C_R} f(z)\,dz \right| \leq \frac{1}{R^4-1} \cdot \pi R \longrightarrow 0 \quad \text{as } R \to \infty.$$

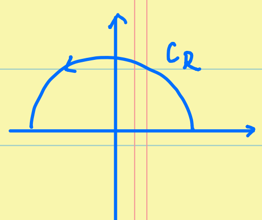
Combine : Let $R \to \infty$ in $(*)$.

Then

$$\frac{\sqrt{2}}{2}\pi = \int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = 2\int_0^{\infty} \frac{1}{1+x^4}\,dx.$$

So $\displaystyle\int_0^{\infty} \frac{1}{1+x^4}\,dx = \frac{\sqrt{2}}{4}\pi.$

<u>RMK.</u> In general, when compute $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{q(x)}\,dx$ for polynomials with $\deg P(x) + 2 \le \deg q(x)$, we have the estimate



$$\int_{C_R} \frac{P(z)}{q(z)}\,dz \longrightarrow 0 \quad \text{as} \quad R \to \infty.$$

<u>Sketch of the proof.</u>

$P(z) = a_0 + a_1 z \cdots + a_m z^m \qquad a_m \ne 0$

$q(z) = b_0 + b_1 z \cdots + b_n z^n. \qquad b_n \ne 0.$

$m + 2 \le n.$

$|P(z)| \sim |a_m|\,|z^m| = |a_m| R^m \quad \text{as} \quad R \to \infty$

$|q(z)| \sim |b_n|\,|z^n| = |b_n| R^n \quad \text{as} \quad R \to \infty.$

Thus, $\displaystyle\left|\frac{P(z)}{q(z)}\right| \sim \left|\frac{a_m}{b_n}\right| \cdot R^{m-n} \quad \text{as} \quad R \to \infty.$

$$\left|\int_{C_R} \frac{P(z)}{q(z)}\,dz\right| \le \left|\frac{a_m}{b_n}\right| R^{m-n} \cdot \pi R$$

$$\le \pi \left|\frac{a_m}{b_n}\right| R^{-1} \longrightarrow 0 \quad \text{as} \quad R \to \infty.$$

$m-n+1 \le -1.$