19. Residues at Poles

The 3 types of isolated singularities

z_0 isolated singularity of $f(z)$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{Principal part of } f},$$

$0 < |z-z_0| < R_2$.

3 possibilities:

(a) Removable singularity: principal part $= 0$, i.e $b_n = 0$ for all $n = 1, 2, 3, \ldots$.

(b) Essential singularity: infinitely many of the b_n are nonzero.

(c) Pole: principal part doesn't vanish, but only finitely many of the b_n are nonzero.

$$\rightsquigarrow \sum_{n=1}^{\infty} b_n (z-z_0)^n = \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

$b_m \neq 0$, $b_j = 0 \ \forall j > m$.

$\rightsquigarrow$ "pole of order m"

Pole of order 1: "simple pole"

Examples:

① $f(z) = \dfrac{e^z - 1}{z}$, $z = 0$ is a removable singularity.

$$f(z) = \frac{\left(1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right) - 1}{z}$$

$$= 1 + \frac{z}{2} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$$

for $z \in \mathbb{C} - \{0\}$.

$\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$ converges for all $z \in \mathbb{C}$ and defines an entire function.

② $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3!z^3} + \frac{1}{4!z^4} + \cdots$

has an essential singularity at $z=0$.

③ $\dfrac{e^z - 1}{z^2} = \dfrac{1}{z} + \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{(n+2)!}$ has a simple pole at $z=0$.

$(m=1)$

④ $\dfrac{z+i}{z^2+1} = \dfrac{z+i}{(z+i)(z-i)}$ has a removable singularity at $z=-i$ and a simple pole at $z=i$.

⑤ $\dfrac{e^z}{(z-2)^5}$ ← pole of order 5 at $z=2$.

Residues at Poles:

Let $g(z)$ be analytic at 0 with $g(0) \neq 0$,

Then $\frac{g(z)}{z^m}$ has a pole of order m

at 0 (here $m \in \{1, 2, 3, \ldots\}$).

Write $g(z) = \sum_{n=0}^{\infty} a_n z^n$, z near 0.

$$\rightsquigarrow \frac{g(z)}{z^m} = \underbrace{\frac{a_0}{z^m} + \frac{a_1}{z^{m-1}} + \cdots + \frac{a_{m-1}}{z}}_{\text{Principal part}} + a_m + a_{m+1} z + \cdots$$

$$\rightsquigarrow \operatorname*{Res}_{z=0} \frac{g(z)}{z^m} = a_{m-1} \underset{\substack{\uparrow \\ \text{def}^n \text{ of } a_{m-1}}}{=} \frac{g^{(m-1)}(0)}{(m-1)!}.$$

<u>Note:</u> for $m=1$, $\frac{g(z)}{z} = \frac{a_0}{z} + a_1 + a_2 z + \cdots$

and $\operatorname*{Res}_{z=0} \frac{g(z)}{z} = a_0 = g(0)$.

Theorem:

An isolated singularity z_0 of $f(z)$ is

<u>a pole of order m</u>

$\Updownarrow$

$f(z) = \frac{\phi(z)}{(z-z_0)^m}$, ϕ analytic at z_0,

with $\phi(z_0) \neq 0$.

In this case $\operatorname{Res} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$.

Proof: $\Uparrow$ easy, expand $\phi(z)$ about z_0.

$\Downarrow$ $$f(z) = \frac{b_m}{(z-z_0)^m} + \cdots + \frac{b_1}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

with $b_m \neq 0$.

$\rightsquigarrow$ set $$\phi(z) = (z-z_0)^m f(z) \qquad (z \neq z_0)$$
$$= b_m + b_{m-1}(z-z_0) + \cdots + b_1 (z-z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m}$$

and define $\phi(z_0) = b_m$. $\leftarrow$ we have to do this since $f(z)$ is not defined at z_0.

$\rightsquigarrow$ $\phi(z)$ is analytic at z_0,

$\phi(z_0) = b_m \neq 0$ and $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$.

$\square$

Examples:

① $f(z) = \dfrac{z^2+1}{z^2+4}$, isolated singularities at $z = \pm 2i$.

$\uparrow$ poles of order 1

$$f(z) = \frac{z^2+1}{(z-2i)(z+2i)} = \frac{\phi(z)}{z-2i},$$

where $\phi(z) = \dfrac{z^2+1}{z+2i}$ is analytic at $2i$

& $\phi(2i) = \dfrac{-4+1}{2i+2i} = \dfrac{-3}{4i} = \dfrac{3i}{4} \neq 0$.

$\rightsquigarrow$ $\operatorname*{Res}_{z=2i} f(z) = \dfrac{3i}{4}$.

② $f(z) = \dfrac{z^4+1}{(z+i)^3}$, $z=-i$ is a pole of $\underline{\text{order } 3}$.
(since $(-i)^4+1 \neq 0$)

$f(z) = \dfrac{\phi(z)}{(z+i)^3}$, $\phi(z) = z^4+1$,

$\phi(-i) = (-i)^4+1 = 1+1 = 2 \neq 0$.

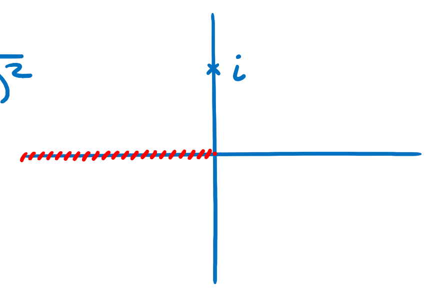
$\rightsquigarrow \operatorname*{Res}_{z=-i} f(z) = \dfrac{\phi''(-i)}{2!}$

$\phi'(z) = 4z^3$, $\phi''(z) = 12z^2$,

$\phi''(-i) = 12(-i)^2 = -12$.

$\rightsquigarrow \operatorname*{Res}_{z=-i} f(z) = -6$.

③ $f(z) = \dfrac{\operatorname{Log} z}{(z-i)^2} = \dfrac{\phi(z)}{(z-i)^2}$



$\phi(i) = \operatorname{Log}(i) = i \cdot \frac{\pi}{2} \neq 0$.

$\rightsquigarrow$ $f(z)$ has a pole of order 2 at $z=i$.

$\rightsquigarrow \operatorname*{Res}_{z=i} f(z) = \phi'(i) = \left.\dfrac{1}{z}\right|_{z=i} = -i$.

④ $f(z) = \dfrac{\sin z}{z^4} = \dfrac{1}{z^4}\sum_{n=0}^{\infty} (-1)^n \dfrac{z^{2n+1}}{(2n+1)!}$

$= \dfrac{1}{z^4}\left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots\right)$

the pole is of order 3 $\underline{\text{not } 4}$. $\longrightarrow$ $= \dfrac{1}{z^3} \boxed{-\dfrac{1}{6z}} + \dfrac{1}{5!}z - \cdots$

$\rightsquigarrow \operatorname*{Res}_{z=0} f(z) = -\dfrac{1}{6}$.

Zeros of analytic functions:

Def^n: If $f(z)$ is analytic at z_0 & $m \geq 1$, $f(z)$ has a zero of order m at z_0

$$\Updownarrow$$

$$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$$

and $f^{(m)}(z_0) \neq 0$.

Examples:

① $f(z) = z^4$ has a zero of order 4 at 0.

② $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$ has a zero of order 1 at 0.

(All the zeros of $\sin z$ have order 1.)

③ $e^z - 1 - z$ has a zero of order 2 at 0.

Theorem: Let $f(z)$ be analytic at z_0.

(i) $f(z)$ has a zero of order m at z_0

$$\Updownarrow$$

(ii) $$f(z) = (z-z_0)^m g(z),$$

$g(z)$ analytic at z_0, $g(z_0) \neq 0$.

Proof: Follows from the Taylor expansion of f at z_0.

(i) $\Rightarrow$ $$f(z) = \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \cdots$$

$$\Rightarrow f(z) = (z-z_0)^m g(z)$$

where $$g(z) = \sum_{n=0}^{\infty} \frac{f^{(n+m)}(z_0)}{(n+m)!} z^n, \quad z \text{ near } z_0.$$

$$\left(g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0 \right)$$

$\Rightarrow$ (ii)

(ii) $\Rightarrow$ $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$

by direct calculation, and similarly

$f^{(m)}(z_0) = m!\, g(z_0)$.

$\Rightarrow$ (i) $\square$

Remark:

Note that the zeros of an analytic function must be isolated (since otherwise one can argue that all the terms in the Taylor expansion vanish and the function is just zero).

Zeros and Poles:

If z_0 is a zero of order m of $f(z)$ then z_0 is a pole of order m of $1/f(z)$.

To see this, write $f(z) = (z-z_0)^m g(z)$
g analytic at z_0, $g(z_0) \neq 0$.

$$\rightsquigarrow \quad \frac{1}{f(z)} = \frac{1}{(z-z_0)^m g(z)} = \frac{\phi(z)}{(z-z_0)^m},$$

$\phi(z) = \frac{1}{g(z)}$ analytic at z_0,

$\phi(z_0) = \frac{1}{g(z_0)} \neq 0$.

$$\rightsquigarrow \quad \operatorname*{Res}_{z=z_0} \frac{1}{f(z)} = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

<u>Example</u>: $f(z) = (z+z^2)^2$, $f(0) = 0$.

$\rightsquigarrow$ $f(z) = z^2(1+z)^2$ so $z=0$ is a zero of order 2.

$\rightsquigarrow$ $h(z) = \frac{1}{f(z)}$ has a pole of order 2.

$$h(z) = \frac{1}{z^2(1+z)^2} = \frac{\phi(z)}{z^2}, \quad \phi(z) = \frac{1}{(1+z)^2}$$

ϕ analytic at 0, $\phi(0) = 1 \neq 0$.

$$\rightsquigarrow \quad \operatorname*{Res}_{z=0} h(z) = \phi'(0) = \left.\frac{-2}{(1+z)^3}\right|_{z=0} = -2.$$

Slightly more generally we have:

Theorem: If $p(z)$ & $q(z)$ are analytic functions such that

(1) $q(z)$ has a zero of order m at z_0;

(2) $p(z_0) \neq 0$;

then $\frac{p(z)}{q(z)}$ has a pole of order m at z_0.

(Proof follows from the above discussion.)

In the case $m=1$ there is a simple formula for $\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)}$.

Theorem: $p(z)$, $q(z)$ analytic. If $q(z)$ has a zero of order 1 at z_0 and $p(z_0) \neq 0$, then $\frac{p(z)}{q(z)}$ has a simple pole at z_0 and

$$\boxed{\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.}$$

Proof: $q(z) = (z-z_0)g(z)$, g analytic at z_0, $g(z_0) \neq 0$.

$\rightsquigarrow \frac{p(z)}{q(z)} = \frac{\phi(z)}{z-z_0}$, $\phi(z) = \frac{p(z)}{g(z)}$

$$\rightsquigarrow \quad \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \phi(z_0) = \frac{p(z_0)}{g(z_0)},$$

but $g(z_0) = q'(z_0)$. $\square$

$$q(z) = (z-z_0)q'(z_0) + \cdots$$
$$= (z-z_0)\,g(z).$$

Example: $f(z) = \cot z = \dfrac{\cos z}{\sin z}$.

$\sin z$ has a zero of order 1 at 0

and $\cos 0 = 1 \neq 0$, so thm applies.

$$\rightsquigarrow \quad \operatorname*{Res}_{z=0} \cot z = \frac{\cos 0}{\cos 0} = 1.$$

By the same argument

$$\operatorname*{Res}_{z=n\pi} \cot z = 1 \quad \leftarrow \quad \frac{\cos n\pi}{\cos n\pi}$$

for any $n \in \mathbb{Z}$.