18. The Residue Theorem

<u>Recall:</u> z_0 is an <u>isolated singularity</u> of $f(z)$

$$\Updownarrow$$

$f(z)$ is analytic in a deleted neighborhood of z_0, but is not defined at z_0.

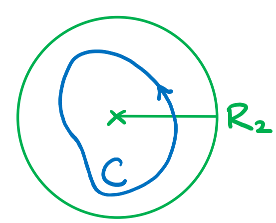
Given $f(z)$ analytic, with isolated singularity at z_0, we can take $R_2 > 0$ small enough such that $f(z)$ is analytic on $0 < |z - z_0| < R_2$.

$\rightsquigarrow$ Laurent expansion:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$= \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \, , \qquad 0 < |z-z_0| < R_2,$$

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz.$$



$$\rightsquigarrow \quad b_1 = c_{-1} = \frac{1}{2\pi i} \int_C f(z) \, dz.$$

<u>Def^n:</u> $f(z)$ analytic, z_0 isolated singularity. We define the <u>residue of f at z_0</u> by

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = c_{-1}.$$

Examples:

① $$e^{1/z} = 1 + \boxed{\frac{1}{z}} + \frac{1}{2z^2} + \frac{1}{3!z^3} + \cdots, \quad z \neq 0.$$

$b_1 = 1$

$$\rightsquigarrow \operatorname*{Res}_{z=0} e^{1/z} = 1.$$

② $$\operatorname*{Res}_{z=0} \frac{1}{z} = 1, \quad \operatorname*{Res}_{z=0} \frac{2+i}{z} = 2+i.$$

③ $$\operatorname*{Res}_{z=0} \frac{1}{z^n} = 0, \quad n \geq 2.$$

④ $$\frac{\cos z}{z} = \boxed{\frac{1}{z}} - \frac{z}{2} + \frac{z^3}{4!} - \frac{z^5}{6!} + \cdots, \quad z \neq 0.$$

$$\rightsquigarrow \operatorname*{Res}_{z=0} \frac{\cos z}{z} = 1$$

⑤ $$\frac{\sin z}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots, \quad z \neq 0.$$

$$\rightsquigarrow \operatorname*{Res}_{z=0} \frac{\sin z}{z^3} = 0.$$

⑥ $$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

For $0 < |z-1| < 1$,

Taylor expansion of $\frac{1}{z-2}$ about $z_0 = 1$.

$$f(z) = \boxed{\frac{1}{z-1}} - \sum_{n=0}^{\infty} d_n (z-1)^n.$$

$$\rightsquigarrow \operatorname*{Res}_{z=1} f(z) = 1.$$

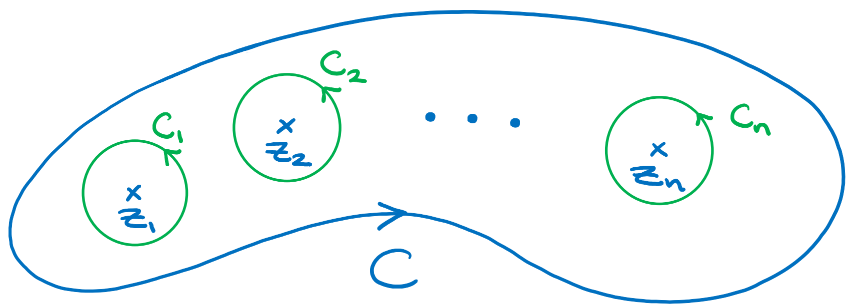
(Similarly $\operatorname*{Res}_{z=2} f(z) = -1.$)

Cauchy's Residue Theorem

C a simple closed contour, positively oriented; $f(z)$ analytic on and inside C, except for a finite number of singular points z_k, $k = 1, 2, \ldots, n$, then

$$\boxed{\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).}$$

Proof:



Define C_k : small circle going counterclockwise around z_k (interior to C & not going around the other z_j).

Since $f(z)$ is analytic in the region inside C and outside the $C_1, \ldots, C_n$, we have

$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz.$$

$$\leadsto \int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z). \quad \square$$

Examples:

① $C : |z| = R > 0$, pos. oriented.

$$\operatorname*{Res}_{z=0} e^{1/z} = 1 \rightsquigarrow \underbrace{\int_C e^{\frac{1}{z}}\,dz = 2\pi i}_{\text{same as } \int_C \frac{1}{z}\,dz}$$

② Same contour.

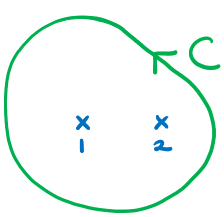
$$e^{1/z^2} = 1 + \frac{1}{z^2} + \frac{1}{2z^4} + \frac{1}{3!z^6} + \cdots$$

no $\frac{1}{z}$ term

$$\operatorname*{Res}_{z=0} e^{1/z^2} = 0 \rightsquigarrow \int_C e^{1/z^2}\,dz = 0.$$

Remark: note that $\int_C 1\,dz = 0$, $\int_C \frac{1}{z^2}\,dz = 0$, $\int_C \frac{1}{2z^4}\,dz = 0, \ldots$

③ $f(z) = \frac{1}{z-1} - \frac{1}{z-2}$, C simple closed, pos. or., with 1 & 2 in interior.



$$\rightsquigarrow \int_C f(z)\,dz = \operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=2} f(z)$$

$$= 1 - 1 = 0.$$

④ (Similar to Example 3 in Section 75 - Residues.)

$C: \quad |z-2|=1$, counterclockwise.

Find $\displaystyle\int_C \frac{dz}{z(z-2)^3}$.

$f(z) = \dfrac{1}{z(z-2)^3}$ analytic on $0<|z-2|<2$,

so we can find a Laurent expansion.

For $0<|z-2|<2$,

$$\frac{1}{z} = \frac{1}{2+z-2} = \frac{1}{2}\cdot\frac{1}{1+\left(\frac{z-2}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(z-2)^n}{2^n},$$

$$\rightsquigarrow \quad f(z) = \frac{1}{z(z-2)^3} = \underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^{n-3}}_{\text{Laurent expansion of } f(z) \text{ in } 0<|z-2|<2.}.$$

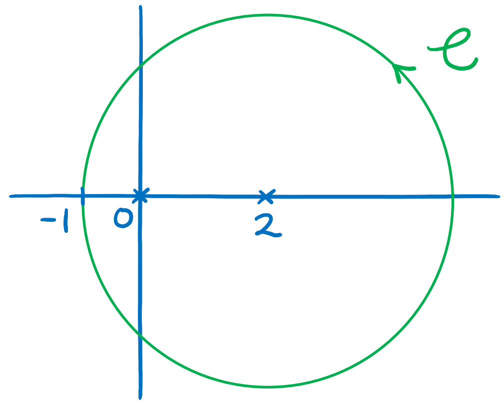
We get $(z-2)^{-1}$ when $n=2$,

$$\rightsquigarrow \quad C_{-1} = \frac{(-1)^2}{2^{2+1}} = \frac{1}{8}.$$

$$\rightsquigarrow \quad \int_C \frac{dz}{z(z-2)^3} = 2\pi i \operatorname*{Res}_{z=2} f(z) = \frac{\pi i}{4}.$$

Now find $\displaystyle\int_{\mathcal{C}} \frac{dz}{z(z-2)^3}$ where $\mathcal{C}$ is

the circle $|z-2|=3$, counterclockwise.



Residue theorem:

$$\int_{\mathcal{C}} \frac{dz}{z(z-2)^3} = 2\pi i \left[\operatorname*{Res}_{z=0} f(z) + \underbrace{\operatorname*{Res}_{z=2} f(z)}_{=\frac{1}{8}} \right].$$

Let $g(z) = \frac{1}{(z-2)^3}$, so $f(z) = \frac{g(z)}{z}$.

$\rightsquigarrow$ $g(z)$ analytic on $|z|<2$, so has Maclaurin series representation

$$g(z) = g(0) + g'(0)z + \frac{g''(0)}{2}z^2 + \dots$$

for $|z|<2$.

$\rightsquigarrow$ $f(z) = \frac{g(z)}{z}$ has Laurent series

$$f(z) = \boxed{\frac{g(0)}{z}} + g'(0) + \frac{g''(0)}{2}z + \dots$$

for $0<|z|<2$.

$\rightsquigarrow$ $\operatorname*{Res}_{z=0} f(z) = g(0) = \frac{1}{(0-2)^3} = -\frac{1}{8}$.

$\rightsquigarrow$ $\int_{\mathcal{C}} \frac{dz}{z(z-2)^3} = 2\pi i \left[-\frac{1}{8} + \frac{1}{8} \right] = 0.$