Ex: Let $f(z) = \overbrace{\frac{1}{z-1}}^{g(z)} - \overbrace{\frac{1}{z-2}}^{h(z)}$

① find $\operatorname{Res}_{z=1} f(z)$

② find $\operatorname{Res}_{z=2} f(z)$

③ find $\int_C f(z)\,dz$

where $C = \{|z| = \frac{3}{2}\}$, "+" oriented

④ find $\int_C f(z)\,dz$

where C: simply closed, "+" oriented, with 1 and 2 inside C.

Hint:

$$\operatorname*{Res}_{z_0}(g+h) = \operatorname*{Res}_{z_0} g + \operatorname*{Res}_{z_0} h$$

why?

Let's say

$$g = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

$$h = \sum_{n=-\infty}^{\infty} \tilde{c}_n (z-z_0)^n$$

$$\Rightarrow \operatorname*{Res}_{z_0} g = c_{-1}, \quad \operatorname*{Res}_{z_0} h = \tilde{c}_{-1}$$

Note:
$$g+h = \sum_{n=-\infty}^{\infty} (c_n + \tilde{c}_n)(z-z_0)^n$$

$$\operatorname*{Res}_{z_0}(g+h) = \text{coefficient of } (z-z_0)^{-1}$$

$$= c_{-1} + \tilde{c}_{-1}$$

$$= \operatorname*{Res}_{z_0} g + \operatorname*{Res}_{z_0} h$$

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① $\operatorname*{Res}_{1} f = \operatorname*{Res}_{1} g + \operatorname*{Res}_{1} h$

Here $\begin{cases} g = \dfrac{1}{z-1} \\ h = -\dfrac{1}{z-2} \end{cases}$

Note: $\operatorname*{Res}_{1} h = 0$

(why? h is analytic 1)

Note: $g = (z-1)^{-1}$

$$= \cdots + 0(z-1)^{-2} + \underbrace{(z-1)^{-1}} + 0(z-1)^{0} + 0(z-1)^{1} + \cdots$$

$$\Rightarrow \operatorname{Res}_{1} g = 1$$

$$\Rightarrow \operatorname{Res}_{1} f = 1 + 0 = 1$$

---

② $\operatorname{Res}_{2} f = \operatorname{Res}_{2} g + \operatorname{Res}_{2} h$

Note: $g = \frac{1}{z-1}$ is analytic at 2

$$\Rightarrow \operatorname{Res}_{2} g = 0$$

Note: $h = -(z-2)^{-1}$

$= \cdots + 0(z-2)^{-2} - (z-2)^{-1} + 0(z-2)^{0} + \cdots$

$\Rightarrow \operatorname{Res}_2 h = -1$

$\Rightarrow \operatorname{Res}_2 f = \operatorname{Res}_2 g + \operatorname{Res}_2 h$

$= 0 - 1 = -1$

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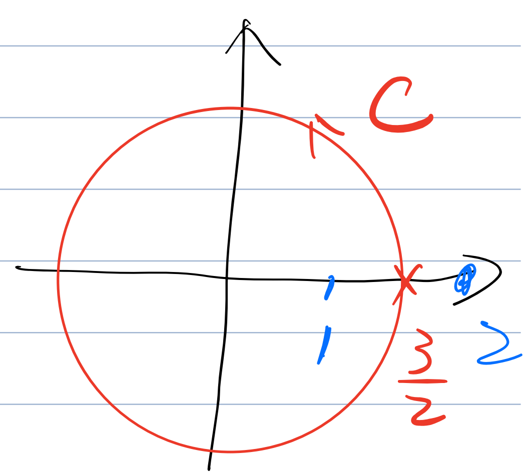
③

By C.R.T



$$\int_C f(z)\,dz = 2\pi i \sum_{z_k \text{ inside } C} \operatorname{Res} f(z_k)$$

$$= 2\pi i \operatorname{Res}_{1} f$$

$$= 2\pi i \cdot 1 = 2\pi i$$

---

④ By C.R.T.,

$$\int_C f(z)\,dz = 2\pi i \sum_{z_k \text{ inside } C} \operatorname{Res}_{z_k} f$$

$$= 2\pi i \left(\operatorname{Res}_{1} f + \operatorname{Res}_{2} f\right) = 2\pi i(1-1)$$

$$= 0$$

We now summarize:

Algorithm of applying Cauchy's Residue Thm:

Q: Compute $\int_C f(z)dz$

where C: simply closed, "+" oriented.

A:

Step 1: find all singularities of f

that are enclosed by C (inside C),

say, there are $z_1, \cdots, z_n$

Step 2: Compute residues of f at $z_1, \cdots, z_n$.

$$\operatorname*{Res}_{z=z_1} f(z), \operatorname*{Res}_{z=z_2} f(z), \cdots, \operatorname*{Res}_{z=z_n} f(z)$$

Add up the above residues and multiply by $2\pi i$ $\Rightarrow$

$$2\pi i\left(\sum_{i=1}^{n} \operatorname*{Res} f(z_i)\right) = 2\pi i\left(\operatorname*{Res}_{z=z_1} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z)\right)$$

Note: The use of Residue Thm to compute integral is very similar to walking a dog.

Compute $\int_C f(z)\,dz$ v.s Walking a dog

① Watch out for the singularities inside the contour C the most common type of singularity is 'pole' ② Add up the residues at these singularities inside C and multiple by $2\pi i$	① Watch out the poles. ② collect dog's residues left at the poles

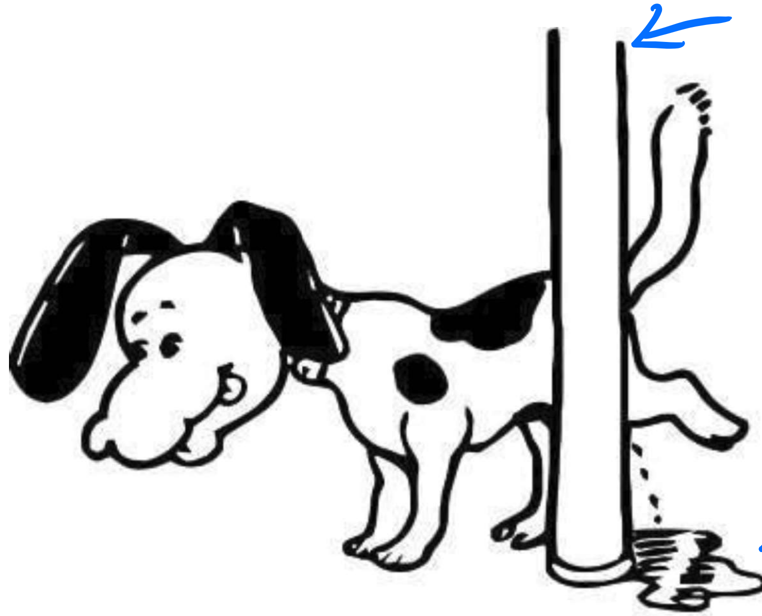
Imagine you are walking a dog in the area enclosed by C

C

Community park

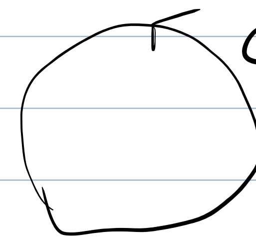
① watch out for 'poles' inside C

② Collect dog's residue



Recall: Cauchy - Goursat thm

- C simply closed, "+" oriented
- f: analytic $\begin{cases} \text{inside } C \\ \text{on } C \end{cases}$



C

$$\Rightarrow \quad \int_C f(z)\,dz = 0$$

why? There's no singularities.

You don't need to collect any residues!

Hence $\int_C f(z)dz = 0$.


Do u know
Residue Thm?

Revisit 3 types of singularity:

z_0: isolated singularity of f

$\Longrightarrow$ f = analytic $\{0 < |z - z_0| < \varepsilon\}$

$$\Longrightarrow \quad f = \underbrace{\sum_{n=0}^{\infty} a_n (z - z_0)^n}_{\text{good part which is defined at } z_0} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}_{\text{bad part which is NOT defined at } z_0}$$

Def^n: we call "$\sum_{n=1} \frac{b_n}{(z - z_0)^n}$" the principal part of f at z_0

3 types of singularity.

(a) Removable singularity: $\Longleftrightarrow$

principal part $= 0$

i.e. $b_1, b_2, b_3 \cdots = 0$

(NOT negative power term)

(b) pole $\Longleftrightarrow$

principal part $\neq 0$, and

only finitly many b_n are

nonzero

(That is,

$$\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$= \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

Here $b_m \neq 0$

After b_m, $b_j = 0$, $j > m$

That is, $\frac{b_m}{(z-z_0)^m}$ is most

negative power term

In this case, we say z_0 is a pole of order m.

(3) essential singularity

$\Longleftrightarrow$ infinitely many $b_n \neq 0$.

($\Longleftrightarrow$ infinitely many negative power terms)

How to compute the Residue of f at z_0?

(a) If z_0 is a removable singularity $\Rightarrow$

$$\operatorname*{Res}_{z_0} f = 0$$

why? $\operatorname*{Res}_{z_0} f =$ Coeff. of $(z-z_0)^{-1}$

$$= 0.$$

(b) pole

We will give an algorithm to compute $\operatorname{Res}_{z_0} f$ for a pole z_0.