17. Taylor and Laurent Series

① Convergence of sequences

- $z_1, z_2, z_3, \dots, \underset{\substack{\| \\ x_n + iy_n}}{z_n}, \dots$ infinite sequence of complex numbers;
- $\underset{\substack{\| \\ x+iy}}{z} \in \mathbb{C}$;

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} |z_n - z| = 0$$

$$\iff |z_n - z| \to 0 \text{ as } n \to \infty$$

$$\iff \lim_{n\to\infty} x_n = x \ \& \ \lim_{n\to\infty} y_n = y.$$

② Convergence of series

$z_1, z_2, z_3, \dots, z_n, \dots$ complex numbers.

$\rightsquigarrow$ "N^{th} partial sum" $\quad S_N = \sum_{n=1}^{N} z_n.$

$\rightsquigarrow$ sequence of partial sums:

$$S_1, S_2, S_3, \dots, S_N, \dots$$

We say that the series $\sum_{n=1}^{\infty} z_n$
"infinite sum"

converges to S if $\lim_{n\to\infty} S_N = S.$

Otherwise, we say $\sum_{n=1}^{\infty} z_n$ diverges.

In other words:

$$\boxed{\sum_{n=1}^{\infty} z_n = \lim_{N\to\infty} \sum_{n=1}^{N} z_n.}$$

Fact:

$$\sum_{n=1}^{\infty} z_n \text{ convergent} \iff \sum_{n=1}^{\infty} x_n \ \& \ \sum_{n=1}^{\infty} y_n \text{ convergent.}$$

$$\boxed{\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n}$$

Absolute Convergence

no cancellation possible

Theorem: If $\sum_{n=1}^{\infty} |z_n|$ converges,
then $\sum_{n=1}^{\infty} z_n$ also converges.

cancellation possible

Def^n: $\sum_{n=1}^{\infty} z_n$ converges absolutely

$\Updownarrow$

$\sum_{n=1}^{\infty} |z_n|$ converges.

$$\boxed{\text{Absolute convergence} \implies \text{convergence.}}$$

Geometric Series: $\sum_{n=0}^{\infty} z^n$, $z \in \mathbb{C}$.

$$S_N = 1 + z + z^2 + \dots + z^N$$

$$z S_N = z + z^2 + \dots + z^N + z^{N+1}$$

$$\rightsquigarrow \quad (1-z) S_N = 1 - z^{N+1}$$

$$\rightsquigarrow \quad S_N = \frac{1 - z^{N+1}}{1-z} \quad \text{provided } z \neq 1.$$

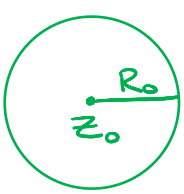
$$\rightsquigarrow \quad \sum_{n=0}^{\infty} z^n \text{ converges} \iff |z| < 1.$$

$$\boxed{\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{provided } |z| < 1.}$$

"radius of convergence"

③ Taylor Series

Taylor's Theorem:



$z_0 \in \mathbb{C}$, $R_0 > 0$, $f(z)$ analytic in the disk $|z - z_0| < R_0$, then

power series representation

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n , \quad |z - z_0| < R_0}$$

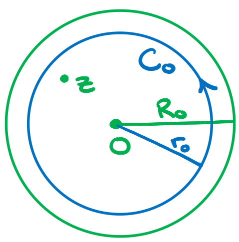
where

$$a_n = \frac{f^{(n)}(z_0)}{n!} , \quad n = 0, 1, 2, \dots .$$

*Maclaurin series when $z_0 = 0$:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \qquad |z| < R_0.$$

Proof for $z_0 = 0$:



Given z with $|z| < R_0$, take $r_0 > 0$ with $|z| < r_0 < R_0$. Take C_0 to be the circle of radius r_0 centered at 0, oriented positively.

Cauchy Integral Formula:

$$\rightsquigarrow \quad f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s-z}\, ds.$$

Write $\displaystyle \frac{1}{s-z} = \frac{1}{s} \cdot \frac{1}{\left(1 - \frac{z}{s}\right)} = \frac{1}{s} \sum_{n=0}^{\infty} \frac{z^n}{s^n} = \sum_{n=0}^{\infty} \frac{z^n}{s^{n+1}}$

(this is valid for $\left|\frac{z}{s}\right| < 1$, so works when $s \in C_0$ & z is interior to C_0). ✓✓

$$\rightsquigarrow \quad f(z) = \frac{1}{2\pi i} \int_{C_0} \sum_{n=0}^{\infty} f(s) \cdot \frac{z^n}{s^{n+1}}\, ds$$

Careful! Interchanging infinite sums & integrals is not valid in general. This needs to be justified.

$$= \sum_{n=0}^{\infty} z^n \underbrace{\frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s^{n+1}}\, ds}_{= a_n}$$

Cauchy I.F. for Derivatives:

$$\rightsquigarrow \quad \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s^{n+1}}\, ds = \frac{f^{(n)}(0)}{n!}. \qquad \square$$

Examples:

① $\frac{1}{1-z}$ is analytic on $\mathbb{C} - \{1\}$,

hence on $\{|z| < 1\}$.

We get the Maclaurin expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.$$

② $\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n, \quad |z| < 1.$

③ $\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$

$$= 1 - z^2 + z^4 - z^6 + \cdots, \quad |z| < 1.$$

entire functions

④ $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}.$

⑤ $\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad z \in \mathbb{C}.$

⑥ $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad z \in \mathbb{C}.$

$$\frac{d}{dz}\left[\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}\right] = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

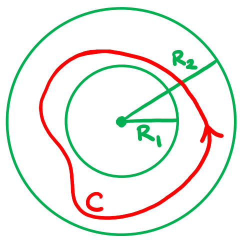
⑦ $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \quad z \in \mathbb{C}.$

⑧ $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \quad z \in \mathbb{C}.$

Recall that $(\cosh z)' = \sinh z$ & $(\sinh z)' = \cosh z$.

④ Laurent Series

Laurent's Theorem:



$z_0 \in \mathbb{C}$, $\quad 0 \le R_1 < R_2$,

$f(z)$ analytic on $D = \{ R_1 < |z - z_0| < R_2 \}$,

C simple closed contour in D, positively oriented, going once around z_0,

then

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz, \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} \, dz, \qquad n = 0, 1, 2, \ldots .$$

Note: a_n & b_n do not depend on the choice of C.

We also write:

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n, \qquad c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz, \qquad n \in \mathbb{Z}.}$$

Examples:

① $e^{1/z}$ on $0 < |z| < \infty$.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}$$

$$\rightsquigarrow \quad e^{1/z} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n}$$

$$= 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots, \quad z \neq 0.$$

$$\left(b_n = \frac{1}{n!}, \quad a_0 = 1, \quad a_n = 0, \quad n = 1, 2, 3, \ldots \right)$$

② $f(z) = \dfrac{1}{(z-i)^3}$ on $0 < |z-i| < \infty$.

This is a trivial example, since $\dfrac{1}{(z-i)^3}$ is already in the form of a Laurent expansion, with $c_n = \begin{cases} 0 & \text{if } n \neq -3 \\ 1 & \text{if } n = -3. \end{cases}$

Exercise: check that for this example we have

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{(z-i)^{n+1}}\, dz = \begin{cases} 0 & \text{if } n \neq -3 \\ 1 & \text{if } n = -3. \end{cases}$$

③ The function $f(z) = \dfrac{1}{z-1} - \dfrac{1}{z-2}$ is analytic on each of the three domains

$D_1 = \{|z| < 1\}$, $D_2 = \{1 < |z| < 2\}$, $D_3 = \{|z| > 2\}$.

Problem: Find the series representation in powers of z for $f(z)$ on each of these three domains.

Domain $D_1 = \{|z|<1\}$:

$$f(z) = -\frac{1}{1-z} + \frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}}$$

$$= -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} - z^n = \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2^{n+1}} - 1\right)}_{=a_n} z^n$$

An ordinary Maclaurin expansion.

Domain $D_2 = \{1<|z|<2\}$:

Key point: we can no longer write $\frac{1}{z-1} = -\frac{1}{1-z}$ as a geometric series in z, since $|z|>1$. But we can write $\frac{1}{1-1/z}$ as a geometric series in $\frac{1}{z}$.

$$f(z) = \frac{1}{z}\cdot\frac{1}{(1-\frac{1}{z})} + \frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}}$$

since $|\frac{1}{z}|<1$ since $|\frac{z}{2}|<1$

$$= \underbrace{\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}}_{=\sum_{n=1}^{\infty}\frac{1}{z^n}} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}, \quad 1<|z|<2.$$

A Laurent expansion!

Domain $D_3 = \{|z| > 2\}$:

$$f(z) = \frac{1}{z} \cdot \frac{1}{(1-\frac{1}{z})} - \frac{1}{z} \cdot \frac{1}{(1-\frac{2}{z})}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} .$$ $\leftarrow$ Laurent expansion!

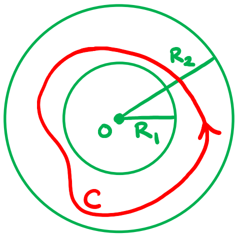
(4) $f(z) = \frac{1}{z(z^2+1)}$ has singularities at $z=0$ and $z = \pm i$. Find the Laurent expansion for $f(z)$ on $\{0 < |z| < 1\}$.

Note: $\frac{1}{z^2+1} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$ for $|z| < 1$.

$$\rightsquigarrow f(z) = \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n z^{2n} = \sum_{n=0}^{\infty} (-1)^n z^{2n-1}$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n z^{2n-1}, \quad 0 < |z| < 1.$$

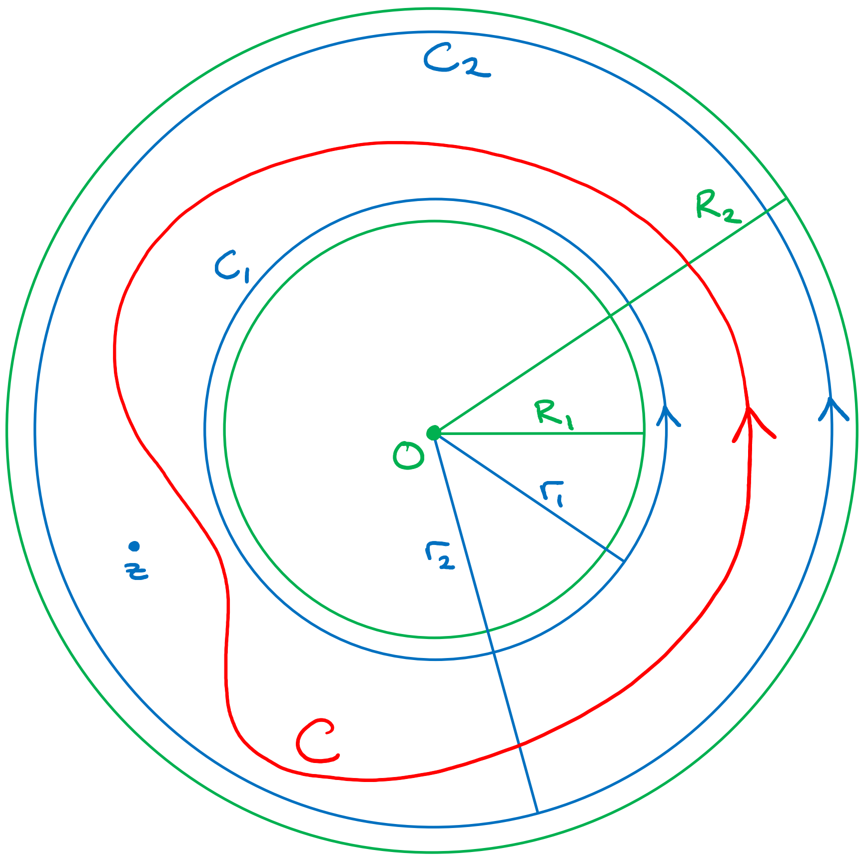
$$= \frac{1}{z} + \sum_{n=0}^{\infty} (-1)^{n+1} z^{2n+1}, \quad 0 < |z| < 1.$$



Proof of Laurent's theorem ($z_0 = 0$):

Given z with $R_1 < |z| < R_2$, take $r_1, r_2 > 0$ with $R_1 < r_1 < |z| < r_2 < R_2$ and with C contained in the annulus with inner radius r_1 & outer radius r_2. Let C_j ($j = 1, 2$) denote the circle of radius r_j, positively oriented and centered at 0.

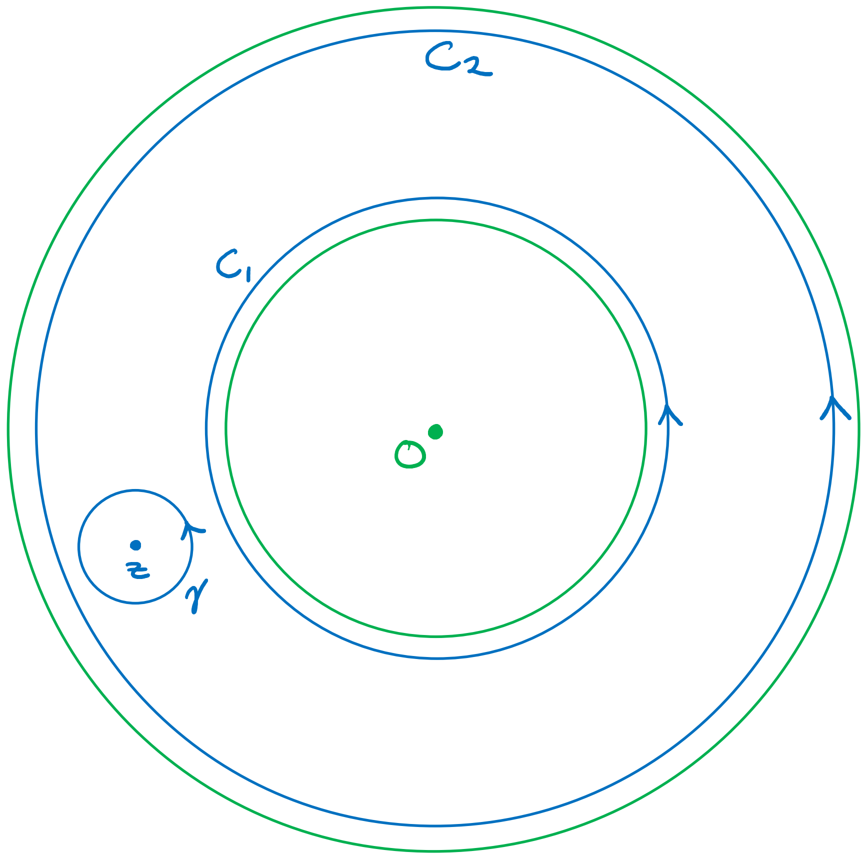
just for convenience



Now consider a small circle γ centered at z and between C_1 & C_2.

The function $f(z)$ is analytic on & interior to γ.

Cauchy Integral Formula: $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(s)}{s - z}\, ds.$



Keyhole surgery gives

$$\int_{C_2} \frac{f(s)}{s-z}\,ds = \int_{C_1} \frac{f(s)}{s-z}\,ds + \int_{\gamma} \frac{f(s)}{s-z}\,ds$$

since C_2 "goes around" C_1 & γ,

and $\frac{f(s)}{s-z}$ is analytic in between.

$$\rightsquigarrow \quad f(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(s)}{s-z}\,ds$$

$$= \underbrace{\frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\,ds}_{\substack{s\in C_2 \text{ so } |s|>|z| \\ \rightsquigarrow \text{ use } \frac{1}{s-z} = \frac{1}{s}\cdot\frac{1}{1-z/s}}} + \underbrace{\frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{z-s}\,ds}_{\substack{s\in C_1 \text{ so } |s|<|z| \\ \rightsquigarrow \text{ use } \frac{1}{z-s} = \frac{1}{z}\cdot\frac{1}{1-s/z}}}$$

For the C_2 integral we use $\frac{1}{s-z} = \sum_{n=0}^{\infty} \frac{z^n}{s^{n+1}}$

as in the proof of Taylor's theorem to get

$$\frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{z-s}\,ds = \frac{1}{2\pi i}\int_{C_2}\sum_{n=0}^{\infty} f(s)\frac{z^n}{s^{n+1}}\,ds$$

Careful! Needs justification.

$$= \sum_{n=0}^{\infty} z^n \underbrace{\frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{s^{n+1}}\,ds}_{=a_n}.$$

For the C_1 integral we have to use

$$\frac{1}{z-s} = \frac{1}{z}\cdot\frac{1}{1-s/z} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{s}{z}\right)^n = \sum_{n=1}^{\infty}\frac{s^{n-1}}{z^n}$$

since $|s|<|z|$ (so $\left|\frac{s}{z}\right|<1$). We get

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{z-s}\,ds = \frac{1}{2\pi i}\int_{C_1}\sum_{n=1}^{\infty} f(s)\frac{s^{n-1}}{z^n}\,ds$$

note that $f(s)s^{n-1} = \dfrac{f(s)}{s^{-n+1}}$

$$= \sum_{n=1}^{\infty}\frac{1}{z^n}\cdot\underbrace{\frac{1}{2\pi i}\int_{C_1} f(s)s^{n-1}\,ds}_{=b_n}.$$

To conclude we simply apply the principle of deformation of paths to our expressions for a_n & b_n to see that in both cases the contour can be taken to be C (instead of C_1 or C_2). This gives the result. $\square$

⑤ Manipulating power/Laurent series

Differentiating term by term:

It is valid to differentiate a convergent power series term by term, provided we stay inside the (open) domain of convergence.

Examples:

① $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad |z| < \infty$

$\rightsquigarrow (\sin z)' = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = \cos z.$

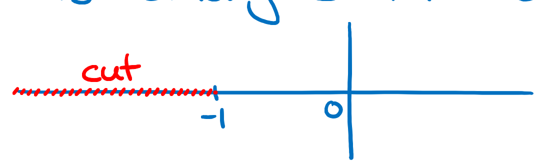
② $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \rightsquigarrow (e^z)' = e^z.$

③ Differentiate $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z|<1$

to get $\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n, \quad |z|<1.$

Integrating term by term: (Valid!)

Example: $\mathrm{Log}(1+z)$ is analytic in the unit disk $|z|<1$.



$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n = 1 - z + z^2 - z^3 + \cdots, \quad |z|<1.$$

$(\mathrm{Log}(1+z))' = \frac{1}{1+z}$ & $\mathrm{Log}(1+0) = 0.$

$$\mathrm{Log}(1+z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1} = z - \frac{z^2}{2} + \frac{z^3}{3} \cdots, \quad |z|<1.$$

Multiplication:

Example: Find the Maclaurin series for $\dfrac{e^z}{z^2+1}$, $|z|<1$.

$$\frac{e^z}{z^2+1} = \left(1+z+\frac{z^2}{2}+\frac{z^3}{3!}+\cdots\right)\left(1-z^2+z^4-z^6+\cdots\right)$$

$$= 1+z+\left(\tfrac{1}{2}-1\right)z^2+\left(\tfrac{1}{6}-1\right)z^3+\left(\tfrac{1}{4!}-\tfrac{1}{2}+1\right)z^4+\cdots$$

Division:

Example: Using $\sinh z = z+\dfrac{z^3}{3!}+\dfrac{z^5}{5!}+\cdots$, $z\in\mathbb{C}$, find the Laurent series expansion for $\dfrac{1}{\sinh z}$ in the punctured disk $0<|z|<\pi$.

$$\rightsquigarrow \quad \sinh z = z\left(\underline{\underline{1}}+\frac{z^2}{3!}+\frac{z^4}{4!}+\cdots\right)$$

$$\rightsquigarrow \quad \frac{1}{\sinh z} = \frac{1}{z}\cdot\frac{1}{1+\frac{z^2}{3!}+\frac{z^4}{5!}+\cdots}$$

$$\begin{array}{r|l}
 & 1-\frac{1}{3!}z^2+\left[\frac{1}{(3!)^2}-\frac{1}{5!}\right]z^4+\cdots \\
\hline
\underline{\underline{1}}+\frac{1}{3!}z^2+\frac{1}{5!}z^4+\cdots & \boxed{1} \\
 & 1+\frac{1}{3!}z^2+\frac{1}{5!}z^4+\cdots \\
\cline{2-2}
 & \boxed{-\frac{1}{3!}z^2}-\frac{1}{5!}z^4-\cdots \\
 & -\frac{1}{3!}z^2-\frac{1}{(3!)^2}z^4-\cdots \\
\cline{2-2}
 & \boxed{\left[\frac{1}{(3!)^2}-\frac{1}{5!}\right]z^4}+\cdots \\
 & \vdots
\end{array}$$

$$\rightsquigarrow \quad \frac{1}{\sinh z} = \frac{1}{z}-\frac{1}{3!}z+\left[\frac{1}{(3!)^2}-\frac{1}{5!}\right]z^3+\cdots,$$

$$0<|z|<\pi.$$