16. Liouville's Theorem and the Fundamental Theorem of Algebra

Recall Cauchy's Inequalities:

- $z_0 \in \mathbb{C}$, $R > 0$, C_R circle centered at z_0 of radius R, oriented positively.
- $f(z)$ analytic on and inside C_R.
- $|f(z)| \leq M_R$ for all $z \in C_R$.

$$\rightsquigarrow \quad |f^{(n)}(z_0)| \leq \frac{n!\, M_R}{R^n}$$

Proof: Apply Cauchy Integral Formula for Derivatives & standard estimate for $\left|\int_{C_R} \cdots \, dz\right|$.

Using this we can prove:

Liouville's Theorem:

If $f(z)$ is entire and bounded, then $f(z)$ is constant.

"A bounded entire function is constant."

Remarks:

① Recall that $f(z)$ entire means $f(z)$ is analytic on the entire complex plane.

② Recall that $f(z)$ bounded means there is $M > 0$ such that $|f(z)| < M$ for all z in the domain of f (here $\mathbb{C}$).

Proof of Liouville's Theorem:

Fix a point $z_0 \in \mathbb{C}$ and apply Cauchy's Inequalities (a consequence of the Cauchy Integral Formula for Derivatives) for $n=1$ and any $R > 0$.

Since $f(z)$ is bounded we have $M > 0$ s.t. $|f(z)| \leq M$ for all $z \in \mathbb{C}$, so we may take $M_R = M$ (independent of R!).

Cauchy's inequality ($n=1$): $\underbrace{|f'(z_0)|}_{\text{constant}} \leq \underbrace{\frac{M}{R}}_{\to 0 \text{ as } R \to \infty}$.

Since this holds for all R & $\frac{M}{R} \to 0$ as $R \to \infty$, we must have $f'(z_0) = 0$.

But z_0 was an arbitrary point in $\mathbb{C}$, and hence the same argument applies to any $z_0 \in \mathbb{C}$ to give $f'(z_0) = 0$.

$\rightsquigarrow$ $f'(z) = 0$ for all $z \in \mathbb{C}$.

Since $\mathbb{C}$ is a domain (in particular, is <u>connected</u>) this implies that f is constant, i.e. there is a constant $c \in \mathbb{C}$ s.t.

$$f(z) = c \quad \text{for all } z \in \mathbb{C}. \qquad \square$$

<u>Remark</u>: Another way of stating Liouville's theorem would be: "<u>Any nonconstant entire function is unbounded.</u>"

We can see this if we look at some familiar examples:

- e^z is unbounded
- $\sin z$ is unbounded (recall that $|\sin z|$ grows as you increase the imaginary part of z).

Application: Fundamental Theorem of Algebra

Theorem (F.T.A.):

Let $P(z)$ be a polynomial of degree n,

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

$n \geq 1$, $a_j \in \mathbb{C}$, $a_n \neq 0$.

Then $P(z)$ has at least one zero in $\mathbb{C}$.

i.e. there is at least one point $z_0 \in \mathbb{C}$ s.t. $P(z_0) = 0$.

Remark: Note that this is not true for real polynomials, e.g., x^2+1 is always positive (for x real) so has no zero in $\mathbb{R}$.

Proof of F.T.A. (proof by contradiction):

We suppose, with a view to obtaining a contradiction, that $P(z)$ has no zero in $\mathbb{C}$.

Then $f(z) = \dfrac{1}{P(z)}$ is an entire function.

Claim: $f(z)$ is bounded on $\mathbb{C}$.

If the claim is true then, by Liouville's theorem, $f(z)$ must be constant.

$$\rightsquigarrow f(z) = c, \quad P(z) = \frac{1}{c}.$$

But this is impossible since the degree of $P(z)$ was $n \geq 1$, meaning that $P(z)$ is not constant. This gives a contradiction provided the claim holds.

<u>Proof of claim:</u> We want to show that if $P(z)$ has no zero then $f(z) = \frac{1}{P(z)}$ is bounded.

$\rightsquigarrow$ Write

$$P(z) = z^n \Big(a_n + \underbrace{\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}}_{=w} \Big)$$

$$= z^n (a_n + w).$$

Note that $w \to 0$ as $z \to \infty$, so $|w| < \frac{|a_n|}{2}$ for $|z| \geq R$ (for some sufficiently large $R > 0$).

Now $|P(z)| \geq |z|^n (|a_n| - |w|)$,

so $|P(z)| \geq R^n \left(|a_n| - \frac{|a_n|}{2} \right) = \frac{|a_n|}{2} R^n$ for all $|z| \geq R$.

a constant

a fixed number

$\rightsquigarrow$ $|f(z)| = \left| \frac{1}{P(z)} \right| \leq \underbrace{\frac{2}{|a_n| R^n}}_{\text{a constant}}$ for $|z| \geq R$.

fixed

$\rightsquigarrow$ $f(z)$ is bounded outside $\{ |z| < R \}$.

But $f(z)$ is continuous on the closed & bounded set $\{|z| \leq R\}$, hence $f(z)$ is bounded on $\{|z| \leq R\}$.

$\leadsto$ $f(z)$ is bounded on $\{|z| \geq R\}$ <u>and</u> on $\{|z| \leq R\}$.

$\leadsto$ $f(z)$ is bounded on $\mathbb{C}$. $\square$

<u>Corollary:</u> If $P(z)$ is as in the F.T.A. then $P(z)$ may be written

$$P(z) = a_n (z - z_1)(z - z_2) \cdots (z - z_n)$$

where $z_1, \ldots, z_n \in \mathbb{C}$ are not necessarily distinct.

<u>Examples:</u>

① $z^2 + 1 = (z - i)(z + i)$

② $z^3 + 2z^2 - (1+2i)z - (2+4i) = (z-i)^2(z+2)$