Math 120A Complex Analysis Midterm II Exam February 28th, 2024

Name (PRINT): Key to Exam 2 PID:

Signature: Section:

Important Instructions: No books, notes, cheat sheets, calculators, cell phones, or any other electronic devices may be used during the exam. Do not start the exam until instructed to do so. You cannot use any results in your homework unless you reprove it.

Problem	Points	Score
#1	9	
#2	16	
#3	10	
#4	15	
#5	3	
Total	50	

The exam starts here: The exam has 53 points including the bonus points, but we will grade it out of 50 points. That means we will add up all the points that you get, but the maximum total points you can have is 50.

(9) **1.** Write $z = x + iy$ with $x, y \in \mathbb{R}$. Let $f(z) = y + ixy$. Find all points in $\mathbb{C}$ where f is complex differentiable and find the value of f' at such point(s). Then find where f is analytic. Show your work.

$f(z) = y + ixy$

$u = y$

$v = xy$

$u_x = 0 \qquad v_x = y$

$u_y = 1 \qquad v_y = x$

$\therefore$ f is complex differentiable at z

$\Leftrightarrow \quad u_x = v_y \qquad u_y = -v_x$

$\Leftrightarrow \quad x = 0 \qquad y = -1$

At $z = (0,-1) \qquad f'(z) = -i$

Since f is not complex differentiable on any open neighbourhood, f is nowhere analytic

(16) **2** Let $u(x, y) = 4xy - 2y$.

(4) (a) Prove u is harmonic on $\mathbb{R}^2$.

$u_x = 4y \quad u_y = 4x - 2$

$u_{xx} = 0 \quad u_{yy} = 0$

$\therefore \; u_{xx} + u_{yy} = 0$

(10) (b) Find all harmonic conjugates v of u on $\mathbb{R}^2$. Show your work.

Suppose v is a harmonic conjugate

$v_x = -u_y = 2 - 4x$

$v_y = u_x = 4y$

$\therefore \; v = \int 4y \, dy = 2y^2 + \phi(x)$

$\therefore \; v_x = \phi'(x) = 2 - 4x$

$\Rightarrow \; \phi(x) = 2x - 2x^2 + C$

$\therefore \; v = 2y^2 + 2x - 2x^2 + C$

(2) (c) Let v be your answer in (b). Find $f(z)$ such that $f = u + iv$. Express the function f in terms of z. Show your work.

$f(z) = 4xy - 2y + i(2y^2 + 2x - 2x^2 + C)$

$= -2i(x^2 - y^2 + i2xy) + 2i(x + iy) + C'$

$= -2i z^2 + 2iz + C'$

(2) **3 (a).** State the definition of the complex logarithmic function $\log z$. Then state the definition of the complex power function z^c, where $c \in \mathbb{C}$.

$\log z = \ln|z| + i \arg z$ where $\arg z$ is a branch of the argument

$z^c = e^{c \log z}$ where $\log z$ is a branch of the logarithm.

(4) **3 (b).** Use definitions to compute $\log(1+i)$ and $\text{Log}(1+i)$. Show your work.

$$\log(1+i) = \ln|1+i| + i\arg(1+i)$$
$$= \frac{1}{2}\ln 2 + i\left(\frac{\pi}{4} + 2k\pi\right)$$

$$\text{Log}(1+i) = \frac{1}{2}\ln 2 + \frac{i\pi}{4}$$

(4) **3 (c).** Use definitions to compute $(-i)^{\frac{2i}{\pi}}$ and P.V.$(-i)^{\frac{2i}{\pi}}$. Show your work.

$$(-i)^{2i/\pi} = e^{2i/\pi \log(-i)}$$
$$= e^{2i/\pi (\ln|-i| + i\arg(-i))}$$
$$= e^{2i/\pi (\ln 1 + i(\frac{-\pi}{2} + 2k\pi))}$$
$$= e^{1-4k}$$

$$\text{P.V. } (-i)^{2i/\pi} = e$$

(2) **4 (a).** State the definitions of $\sin z$ and $\cos z$, with $z \in \mathbb{C}$.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

(5) **4 (b). Use definitions of** $\cos z$ to prove $\cos(\pi + z) = -\cos z$, where $z \in \mathbb{C}$. (You must use the definition to prove it. You will get no points if you use other formulas instead of the definition. Hint: $\cos \pi = -1$).

$$\cos(\pi + z) = \frac{e^{i(z+\pi)} + e^{-i(z+\pi)}}{2}$$

$$= \frac{e^{iz} e^{i\pi} + e^{-iz} e^{-i\pi}}{2}$$

$$= -\frac{e^{iz} + e^{-iz}}{2}$$

$$= -\cos z$$

(8) **4 (c).** Determine whether the following statements are true or false. Write "True" or "False" in the parentheses. **You don't need to justify.**

(1). If v is a harmonic conjugate of u on $\mathbb{R}^2$, then $-u$ is a harmonic conjugate of v on $\mathbb{R}^2$. (True)

(2). For every complex number z, we have $\sin^2 z + \cos^2 z = 1$. (True)

(3). The function Log$z$ is entire. (False)

(4). It holds that $\text{Log}(z^3) = 3\text{Log}z$ for every nonzero complex number z. (False)

Bonus Questions.

(1.5) **5 (a).** Do you have any suggestions to improve the quality of future classes? Whatever suggestions you make, you will get the 1.5 points for this problem. Your suggestions are highly appreciated.

Thank you very much for all your suggestions!

(1.5) **5 (b).** Find all complex numbers z such that $\sin z = \cos z$. Justify your answer. No points will be given if there is only an answer without a proof.

$\sin z = \cos z$

$\frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2}$

$\Rightarrow \quad e^{2iz} - 1 = i e^{2iz} + i$

$\Rightarrow \quad e^{2iz} = \frac{1+i}{1-i} = i$

$\Rightarrow \quad 2iz = \log i = \ln|i| + i\arg(i) = i\left(\frac{\pi}{2} + 2k\pi\right)$

$\Rightarrow \quad z = \frac{\pi}{4} + k\pi$

The exam ends here.

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At $z = (1, 0) \qquad f'(z) = 1$

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$\therefore v = -\int 4x \, dx = -2x^2 + \phi(y)$

$\therefore v_y = \phi'(y) = 4y + 2$

$\Rightarrow \phi(y) = 2y^2 + 2y + c$

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(2) (c) Let v be your answer in (b). Find $f(z)$ such that $f = u + iv$. Express the function f in terms of z. Show your work.

$f(z) = 4xy + 2x + i(-2x^2 + 2y^2 + 2y + C)$

$= -2i(x^2 - y^2 + i2xy) + 2(x + iy) + c'$

$= -2i z^2 + 2z + c'$

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$$= e^{(1-4k)/2}$$

$$\text{P.V. } (-i)^{i/\pi} = e^{1/2}$$

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