1. $S = \{z \mid 0 < |z-1| < 1\} \cup \{-i\}$

a) $\text{int}(S) = \{z \mid 0 < |z-1| < 1\}$

$-i$ is not in the interior since it is not contained in a disc which is a subset of S.

b) $\text{ext}(S) = \{|z-1| > 1\} \cap \{z \neq -i\}$

1 is not in the exterior for the same reason as above.

c) $\text{bdry}(S) = \{|z-1| = 1\} \cup \{1\} \cup \{-i\}$

d) $\text{cl}(S) = \{|z-1| \leq 1\} \cup \{-i\}$

e) $\text{Acc}(S) = \{|z-1| \leq 1\}$

$-i$ is not an accumulation point.

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i is not an accumulation point.

2.(a) $z = \frac{3-i}{1+i}$

$= \frac{3-i}{1+i} \frac{1-i}{(1-i)}$

$= \frac{2-4i}{2}$

$= 1-2i$

$\bar{z} = 1+2i$

$|z| = \sqrt{1^2+(-2)^2}$

$= \sqrt{5}$

$z = \frac{3+i}{1-i}$

$= \frac{3+i}{1-i} \frac{1+i}{(1+i)}$

$= \frac{2+4i}{2}$

$= 1+2i$

$\bar{z} = 1-2i$

$|z| = \sqrt{1^2+(2)^2}$

$= \sqrt{5}$

2(b) $3i = 3e^{i\pi/2}$

$= 3e^{i(\pi/2 + 2k\pi)}$

$\therefore$ the 6^{th} roots are given by $3^{1/6} e^{i(\pi/12 + k\pi/3)}$ $k = 0, 1, \ldots, 5$

the 8^{th} roots are given by $3^{1/} e^{i(\pi/16 + k\pi/4)}$ $k = 0, 1, \ldots, 7$

3(ii) We show that $\lim_{z \to 1} (z + 2\bar{z}) = 3$

for every $\varepsilon > 0$ pick $\delta = \varepsilon/3$

so that whenever $|z-1| < \delta$

$$|z + 2\bar{z} - 3| = |(z-1) + 2(\bar{z}-1)|$$
$$\leq |z-1| + 2|\bar{z}-1|$$
$$= |z-1| + 2|z-1|$$
$$= 3|z-1|$$
$$< 3\delta = \varepsilon$$

We show that $\lim_{z \to 1} (2z + \bar{z}) = 3$

for every $\varepsilon > 0$ pick $\delta = \varepsilon/3$

so that whenever $|z-1| < \delta$

$$|2z + \bar{z} - 3| = |2(z-1) + (\bar{z}-1)|$$
$$\leq 2|z-1| + |\bar{z}-1|$$
$$= 2|z-1| + |z-1|$$
$$= 3|z-1|$$
$$< 3\delta = \varepsilon$$

3(b) $f(z) = |z|^2 + 1$

We have $\displaystyle\lim_{\Delta z \to} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\overline{z}_0 + \overline{\Delta z}) - z_0 \overline{z}_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0 \overline{\Delta z} + \Delta z\, \overline{z}_0 + \Delta z\, \overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \quad z_0 \frac{\overline{\Delta z}}{\Delta z} + \overline{z}_0 + \overline{\Delta z}$$

Now $\displaystyle\lim_{\Delta z \to 0} \overline{z}_0 = \overline{z}_0$ $\displaystyle\lim_{\Delta z \to 0} \overline{\Delta z} = 0$

$\therefore$ $\displaystyle\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ exists iff $\displaystyle\lim_{\Delta z \to 0} z_0 \frac{\overline{\Delta z}}{\Delta z}$ exists

Claim : $\displaystyle\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$ does not exist

Using the claim, $f'(z)$ only exists if $z_0 = 0$.

Proof of claim:

Suppose $\Delta z = \Delta x$ real. Then $\displaystyle\lim_{\Delta x \to 0} \frac{\overline{\Delta x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$

$\Delta z = i\Delta y$ imaginary then $\displaystyle\lim_{\Delta y \to 0} \frac{\overline{i\Delta y}}{i\Delta y} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$

Since the directional limits do not agree, the limit cannot exist.

4. $w = z^2$

write $z = x + iy$ $\quad w = u + iv$

$\therefore$ $u + iv = x^2 - y^2 + i(2xy)$

$u + iv \in S \iff u = \operatorname{Re} w = 1$

$\iff x^2 - y^2 = 1$

this is a hyperbola

5. We show that $\lim_{z \to i} (z + i)^2 = -4$

for every $\varepsilon > 0$ pick $\delta = \min\left\{ \frac{\varepsilon}{5}, 1 \right\}$

so that whenever $|z - i| < \delta$

$$
\begin{aligned}
|(z+i)^2 - (-4)| &= |(z+i)^2 + 4| \\
&= |z^2 + 2zi - 1 + 4| \\
&= |z^2 + 2zi + 3| \\
&= |(z-i)^2 + 4zi + 4| \\
&= |(z-i)^2 + 4i(z-i)| \\
&\le |z-i|^2 + 4|z-i| \\
&< \delta^2 + 4\delta \\
&< 5\delta \\
&< \varepsilon
\end{aligned}
$$