1. $u(x,y) = x^2 - y - y^2$

a) $u_x = 2x \quad u_y = -1-2y$

$u_{xx} = 2 \quad u_{yy} = -2$

$\therefore \quad u_{xx} + u_{yy} = 0$

b) Suppose v is a harmonic conjugate

$v_x = -u_y = +1+2y$

$v_y = u_x = 2x$

$\therefore \quad v = \int 2x \, dy = 2xy + \phi(x)$

$v_x = 2y + \phi'(x) = 1+2y$

$\Rightarrow \quad \phi'(x) = 1$

$\Rightarrow \quad \phi(x) = x + C$

$\therefore \quad v = 2xy + x + C$

c) $f(z) = x^2 - y - y^2 + i(2xy + x + C)$

$= x^2 - y^2 + i2xy - y + ix + C'$

$= (x+iy)^2 + i(x+iy) + C'$

$= z^2 + iz + C'$

2. $\log(1-i) = \ln|1-i| + i\arg(1-i)$

$= \frac{1}{2}\ln 2 + i\left(-\frac{\pi}{4} + 2k\pi\right)$

$\mathrm{Log}(1-i) = \ln|1-i| + i\,\mathrm{Arg}(1-i)$

$= \frac{1}{2}\ln 2 - \frac{i\pi}{4}$

3. $(-i)^{(-i)} = e^{-i\log(-i)}$

$= e^{-i(\ln|-i| + i\arg(-i))}$

$= e^{-i(\ln 1 + i(-\pi/2 + 2k\pi))}$

$= e^{-\pi/2 + 2k\pi}$ $\quad k = 0, \pm 1, \ldots$

P.V. $(-i)^{(-i)} = e^{-\pi/2}$

4. $i^{(1-i)} = e^{(1-i)\log i}$

$= e^{(1-i)(\ln|i| + i\arg(i))}$

$= e^{(1-i)i(\pi/2 + 2k\pi)}$

$= e^{i(\pi/2 + 2k\pi) + \pi/2 + 2k\pi}$

$= e^{\pi/2 + 2k\pi}\, e^{i(\pi/2 + 2k\pi)}$

$= i e^{\pi/2 + 2k\pi}$ $\quad k = 0, \pm 1, \ldots$

P.V. $i^{(1-i)} = i e^{\pi/2}$

5. $$\sin\left(z+\frac{\pi}{2}\right) = \frac{e^{i(z+\frac{\pi}{2})} - e^{-i(z+\pi/2)}}{2i}$$

$$= \frac{e^{i\pi/2}e^{iz} - e^{-i\pi/2}e^{-iz}}{2i}$$

$$= \frac{ie^{iz} + ie^{-iz}}{2i}$$

$$= \cos z$$

$$\sin^2 z + \cos^2 z = \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2$$

$$= \frac{(e^{iz}+e^{-iz})^2 - (e^{iz}-e^{-iz})^2}{4}$$

$$= \frac{(e^{iz}+e^{-iz}+e^{iz}-e^{-iz})(e^{iz}+e^{-iz}-e^{iz}+e^{-iz})}{4}$$

$$= \frac{2e^{iz}\,2e^{-iz}}{4}$$

$$= 1$$