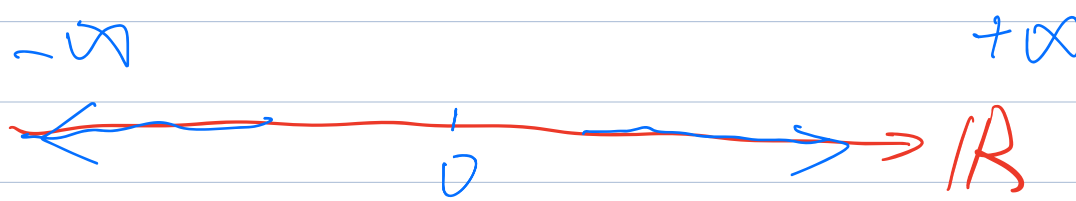
Limits involving ∞.

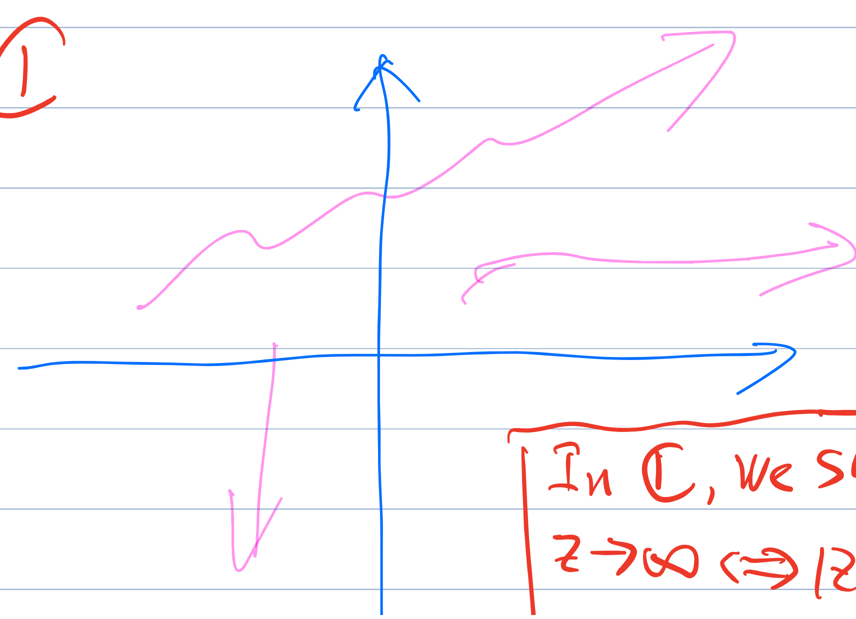
Recall in Calculus we have.

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty \ ; \ \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x\to +\infty} \frac{1}{x} = 0 \ ; \ \lim_{x\to -\infty} \frac{1}{x} = 0$$



In $\mathbb{C}$



In $\mathbb{C}$, we say

$z \to \infty \iff |z| \to +\infty$

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In complex analysis, we also have limits involving ∞:

Def^n: means $|f(z)| \to +\infty$

① $\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$

② $\lim_{z \to \infty} f(z) = w_0 \iff \lim_{w \to 0} f(\frac{1}{w}) = w_0$

③ $\lim_{z \to \infty} f(z) = \infty \iff \lim_{w \to 0} \frac{1}{f(\frac{1}{w})} = 0$.

How ~~to~~ remember them?

"$\frac{1}{\infty} = 0$" and "$\frac{1}{0} = \infty$".

E.g Let $f(z) = \dfrac{5}{z-i}$

$$\lim_{z \to i} f(z) = \ ?$$

continuous

Note: $\dfrac{1}{f(z)} = \dfrac{z-i}{5}$ satisfies

$$\lim_{z \to i} \frac{1}{f(z)} = \frac{1}{f(i)} = \frac{i-i}{5}$$
$$= 0$$

Hence $\displaystyle\lim_{z \to i} f(z) = \infty$

" $\dfrac{1}{0} = \infty$; $\dfrac{1}{\infty} = 0$ "

Joke time:

Q: prove

"$\frac{1}{\infty} = 0$" $\Rightarrow$ "$\frac{1}{0} = \infty$"

A: Start with the LHS

$$\frac{1}{\infty} = 0$$

Step 1: Rotate ↶ 90°

$$-18 = 0$$

Step 2: Add 8

$$-10 = 8$$

Step 3: Rotate ↷ 90° $\Rightarrow$ $\frac{1}{0} = \infty$

Now: Today's new topic

Derivative

Again recall Calculus version of "derivative"

(1) $f(x): I \subseteq \mathbb{R} \to \mathbb{R}$, $x_0 \in I$.

We say f is real differentiable at x_0

$\iff$ $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists

write $x = x_0 + \Delta x$

$$= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Here $\Delta y = f(x_0 + \Delta x) - f(x_0)$

Derivative in complex analysis is defined very similarly:

Defn: Let $f: S \subseteq \mathbb{C} \to \mathbb{C}$

f is (complex) differentiable at z_0

$\Longleftrightarrow$

$$(*) \quad f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists}$$

exists as a complex number

Remark: ① In the furture, if we simply say "differentiable", then we shall understand as "complex differentiable"

② one can also write $(*)$ as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

write

$z = z_0 + \Delta z$

$\Delta w = f(z) - f(z_0)$

We often do NOT emphasize a specific pt z_0, and just consider a general pt z.

Then we write:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

E.g.: If f differentiable? If so, find f'.

① $f(z) = z^2$

We compute

$$\frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$= \frac{z^2 + 2z(\Delta z) + (\Delta z)^2 - z^2}{\Delta z}$$

$$= 2z + (\Delta z)$$

we take limit $\Rightarrow$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z)$$

$$= 2z + \lim_{\Delta z \to 0} \Delta z = 2z \text{ exists!}$$

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② $f(z) = \bar{z}$

we compute $\dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

$$= \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$

$$= \frac{\overline{\Delta z}}{\Delta z}$$

Take limit:

Consider $\displaystyle\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$

(Then just copy the proof from last lecture to show the limit DNE)

① When $\Delta z \to 0$ along x-axis, $\Delta z = \Delta x + i0 \to 0$

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

② When $\Delta z \to 0$ along y-axis, $\Delta z = 0 + i\Delta y \to 0$

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1.$$

Hence the limit DNE $\Rightarrow$ At every z, $f(z) = \bar{z}$ is NOT diff...

③ $f(z) = |z|^2 = z \cdot \bar{z}$

We compute $\dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

$$= \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$$= \frac{(z+\Delta z)\overline{(z+\Delta z)} - z\bar{z}}{\Delta z}$$

$$= \frac{z \cdot \bar{z} + z\overline{\Delta z} + \Delta z \cdot \bar{z} + \Delta z \overline{\Delta z} - z\bar{z}}{\quad}$$

Δz

$$= z\left(\frac{\overline{\Delta z}}{\Delta z}\right) + \bar{z} + \overline{\Delta z}$$

Take limit:

$$\lim_{\Delta z \to 0}\left(z\left(\frac{\overline{\Delta z}}{\Delta z}\right) + \bar{z} + \overline{\Delta z}\right)$$

$$= z\lim_{\Delta z \to 0}\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \lim_{\Delta z \to 0}\overline{\Delta z}$$

$$= \begin{cases} \bar{z} = 0 & \text{if } z = 0 \\ \text{DNE} & \text{if } z \neq 0 \end{cases}$$

Summary: $f(z) = |z|^2$ is differentiable at $z = 0$ and

$f'(0) = 0$; NOT differentiable at $z \neq 0$.

We make a remark on E.g ③:

Note: write $z = x + iy$

$u + iV = f(z) = |z|^2 = \underbrace{(x^2 + y^2)}_{u} + i\underbrace{0}_{V}$

$\Rightarrow \begin{cases} u = x^2 + y^2 \\ v = 0 \end{cases}$

Warning: $u(x,y), V(x,y)$ are both real differentiable (i.e., $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist)

$\not\Longrightarrow$ $f = u + iv$ complex differentiable

Q: How about the converse direction?

(i.e. $f = u + iv$ complex diff $\overset{???}{\Longrightarrow}$ u, v real differentiable)

A: Spoiler Alert: Yes!

will discuss in the future.

Remark:

An intuitive rule: $\bar{z}$ term is bad when it comes to complex differentiability.

That is, when $\bar{z}$ term appears in $f(z)$ most times f is NOT complex diff......

E.g: $z^5 + z^2 + z + 1$ ✓

$$\begin{cases} \bar{z}^3 + 1 \\ 2\bar{z}^4 + z^2 \end{cases}$$ ✗

Rule of Differentiation.

Assume f, g are differentiable. Then

① $(f+g)'(z) =$

② $(fg)'(z) =$

③ $\left(\frac{f}{g}\right)'(z) =$

④ $\frac{d}{dz}[c]$

$\frac{d}{dz}[z^n] =$

⑤ Chain rule

$\frac{d}{dz}[f \circ g(z)] =$