Limit and Continuity

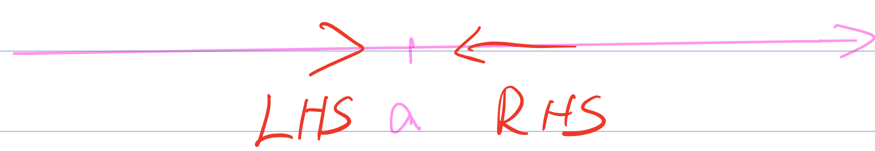
Recall in Calculus:

Let $f: I \to \mathbb{R}$ be a real function, and $I \subseteq \mathbb{R}$ is an interval containing the point $a \in \mathbb{R}$.

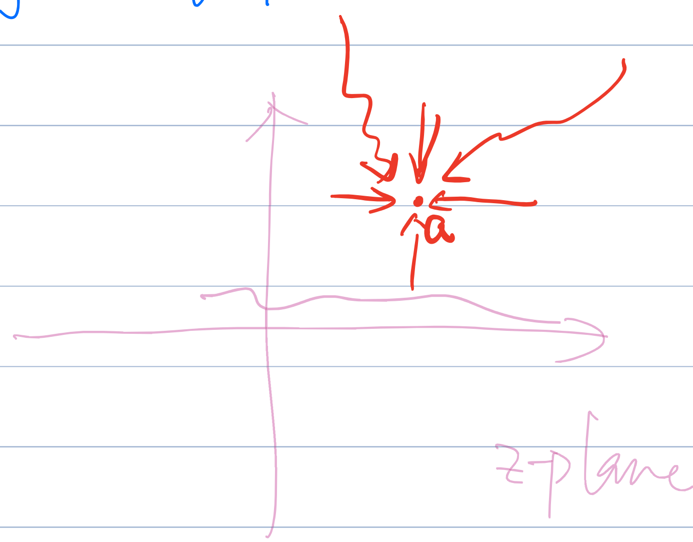
we say $\lim_{x \to a} f(x) = L$ if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

Remark: • In $\mathbb{R}$, x has 2 ways to approach a: from LHS and RHS



In complex analysis, to define limit, we cannot just look



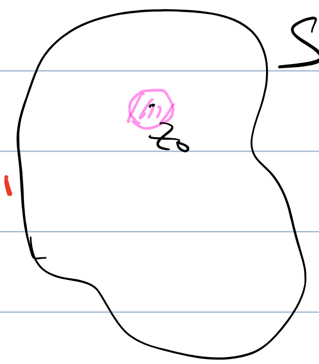
In $\mathbb{C}$, 'Limit exists' requires the limits exist and equal along all possible directions.

Conclusion: we need to use ε–δ language to define limit in $\mathbb{C}$.

Let $S \subseteq \mathbb{C}$, and $f: S \subseteq \mathbb{C} \to \mathbb{C}$ a (complex) function. Let z_0 be an interior pt of S.

Def^n: (Limit in complex analysis)

We say $\lim_{z \to z_0} f(z) = L \iff$



for every $\varepsilon > 0$, we can "find $\delta > 0$" (depends on ε) such that

if $0 < |z - z_0| < \delta$, then $|f(z) - L| < \varepsilon$

Remark:

① It means when z gets close to z_0, (in whichever direction), but not equal to z_0, $f(z)$ gets close to L.

② The limit is unique.

② The Limit is unique!

E.g 1. Find $\lim_{z\to z_0} f(z)$ and give a proof.

(a) $f(z) = 2iz$, $z_0 = 3$.

A: $\lim_{z\to 3} \underbrace{2iz}_{f(z)} = \underbrace{6i}_{L}$

Pf: Given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{2}$,

If $0 < |z - z_0| < \delta$ (i.e $0 < |z-3| < \delta$)

then $|f(z) - L| = |2iz - 6i|$

$= |2i(z-3)|$

$= 2|z-3| < 2\delta = \varepsilon$

we want: find a good δ
s.t
$0 < |z-3| < \delta \Rightarrow$
$2|z-3| < \varepsilon$

(b) $f(z) = \bar{z}$, $\qquad z_0 = a \in \mathbb{C}$.

A: $\displaystyle\lim_{z \to a} \bar{z} = \bar{a}$

Pf: Given any $\varepsilon > 0$, take $\delta = \varepsilon$,

If $0 < |z - a| < \delta$, $\Rightarrow$

$$|f(z) - \bar{a}| = |\bar{z} - \bar{a}| = |\overline{z - a}|$$

$$= |z - a| < \delta = \varepsilon$$

$$\boxed{\text{Hint: } |\bar{w}| = |w|}$$

Hence by defn,

$$\lim_{z \to a} \bar{z} = \bar{a}.$$

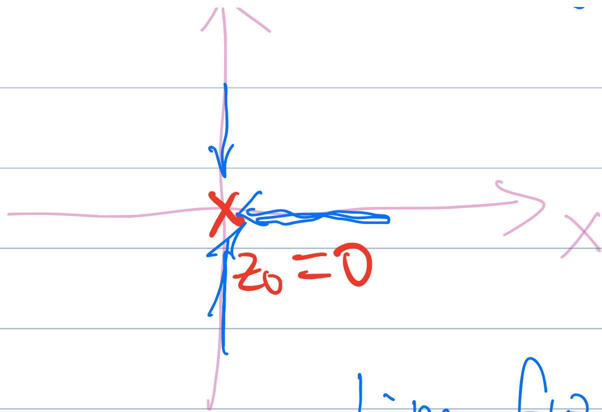
Eg 2 : Let $f(z) = \begin{cases} \dfrac{z}{\bar{z}} & z \in \mathbb{C} - \{0\} \\ 0 & z = 0 \end{cases}$

prove $\lim_{z \to 0} f(z)$ DNE (does NOT exist)

Hint: when we prove $\lim_{z \to z_0} f(z)$ DNE, we often choose two distinct directions (normally horizontal and vertical directions) to $\longrightarrow z_0$ and show we get two distinct limits along the two directions.

Pf: ① when z goes to 0 along x-axis,

$z_0 = 0$ y



$$z = x + iy = x$$

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{z}{\bar{z}}$$

$$= \lim_{z \to 0} \frac{x}{x} = 1$$

② When z goes to 0 along y-axis,

$$z = x + iy = iy$$

$$\Rightarrow \lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{y \to 0} \frac{iy}{-iy}$$

$$= -1$$

Thus we get two different limits along two different directions $\Rightarrow$

$$\lim_{z \to z_0} f(z) \quad \text{DNE}$$

Def^n: (continuity)

Let $f: S \subseteq \mathbb{C} \to \mathbb{C}$, let z_0 be an interior pt of S.

We say f is continuous at z_0

$\iff$ $\begin{cases} ① \lim_{z \to z_0} f(z) \text{ exists} & ? \\ ② \lim_{z \to z_0} f(z) = f(z_0) & ? \end{cases}$

Q: Let $f(z) = \begin{cases} \frac{z}{\bar{z}} & \text{if } z \in \mathbb{C} - \{0\} \\ 0 & \text{if } z = 0 \end{cases}$

Is f continuous at 0?

A: No, as $\lim_{z \to z_0} f(z)$ DNE.

Facts about limits and continuity

(No need to know the proof, but read book to understand)

- If $\lim_{z \to z_0} f(z)$, $\lim_{z \to z_0} g(z)$ exists, then

① $\lim_{z \to z_0} [f(z) + g(z)] = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z)$

② $\lim_{z \to z_0} [f(z) g(z)] = (\lim_{z \to z_0} f(z)) (\lim_{z \to z_0} g(z))$

③ $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)}$

given $\lim_{z \to z_0} g(z) \neq 0$.

• If f, g are continuous at z_0, then

① $f+g$ is continuous at z_0

② fg is continuous at z_0

③ $\frac{f}{g}$ is continuous at z_0 given $g(z_0) \neq 0$.

E.g: ① every polynomial function

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

all $a_j \in \mathbb{C}$, is continuous at every $z_0 \in \mathbb{C}$

② rational functions $\frac{p(z)}{q(z)}$

p, q polynomials

are continuous at z_0 if $q(z_0) \neq 0$

③ $f(z) = \bar{z}$ is continuous at every $z_0 \in \mathbb{C}$.

$$\left(\Longleftrightarrow \quad \lim_{z \to z_0} \bar{z} = f(z_0) = \overline{z_0} \right)$$

$\mathbb{R}^2 \to \mathbb{R}^2$

Thm: Let $f(z): S \subseteq \mathbb{C} \to \mathbb{C}$,

write $f(z) = u(x,y) + i v(x,y)$;

Let $z_0 = x_0 + i y_0 \in S$

Then f is continuous at z_0

$\Leftrightarrow$ $u(x,y)$ and $v(x,y)$ are

continuous at (x_0, y_0)

E.g. e^z is continuous at every $z_0 \in \mathbb{C}$

Note: $e^z = e^{x+iy} = e^x e^{iy}$

$= e^x(\cos y + i \sin y)$

$$= \underbrace{e^x \cos y}_{u} + i \underbrace{e^x \sin y}_{v}$$

Def^n: (composition of functions)

$$f \circ g(z) = f(g(z))$$

given f is defined at $g(z)$

the value of f at $g(z)$

Thm: Continuity of composition functions

The composition of two continuous function is still continuous

E.g e^{3z^2-5} is continuous

composition of $\begin{cases} f(w) = e^{w} \\ g(z) = 3z^2-5 \end{cases}$

Limits involving ∞.

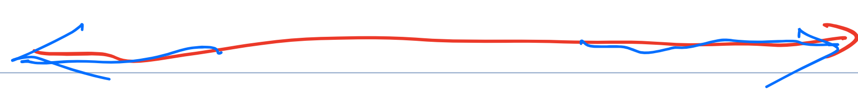
$+\infty$

$-\infty$

Recall in Calculus, we have.

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty \; ; \; \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x\to +\infty} \frac{1}{x} = 0 \; ; \; \lim_{x\to -\infty} \frac{1}{x} = 0$$



$\mathbb{R}$

In complex analysis, we also have limits involving ∞: In $\mathbb{C}$, just call all of them ∞.

Def^n:

①

② $\lim\limits_{z \to \infty} f(z) = w_0 \iff \lim\limits_{w \to 0} f\left(\frac{1}{w}\right) = w_0$

③ $\lim\limits_{z \to \infty} f(z) = \infty \iff \lim\limits_{w \to 0} \dfrac{1}{f\left(\frac{1}{w}\right)} = 0$.

E.g Let $f(z) = \frac{5}{z-i}$

$\lim_{z \to i} f(z) = \ ?$

Joke time:

Q: prove

" $\frac{1}{\infty} = 0$ " $\Rightarrow$ " $\frac{1}{0} = \infty$ "

A: