Review class

Plan: we will do Q3, Q4 on the practice exam (Complex integral problems) first, and then discuss Q2 and finally Q1.

Recall:

Summary for Complex integral problems:

Q: How many methods do we have for computing contour integrals?

A:

① (By definition) If $C: z(t),\ a \le t \le b$

$$\Rightarrow \int_C f(z)\,dz = \int_a^b f(z(t)) \cdot z'(t)\,dt$$

② (Finding antiderivative)

When you compute $\int_C f(z)\,dz$, if you can find F, D s.t

$$\begin{cases} C: \text{ is entirely in } D \\ F: \text{ analytic in } D \\ \quad F' = f \text{ on } D \end{cases}$$

$\Rightarrow \int_C f(z)\,dz = F(z(b)) - F(z(a))$

If in addition, C is closed $\left(z(a) = z(b)\right)$

$\Rightarrow \int_C f(z)\,dz = 0$

③ (Cauchy-Goursat Thm)

If C: simply closed

f: analytic inside and on C

$\Rightarrow \int_C f(z)\,dz = 0$

④ (C.I.F for derivatives)

Thm: If

- C: simply closed. "+" oriented
  f: analytic inside and on C
  z_0: a point inside C

$\Rightarrow$

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0) \quad \star$$

$n = 0, 1, 2, 3, \ldots$

require c to be closed

Method ①, ②, ③, ④

Q: When to use which method?

A: Some tips:

(a) When C is not closed, choose from ①, ②

- If you cannot parametrize C, do NOT use ①
- If you cannot find the anti-derivative of f, do NOT use ②;

(b) When C is simply closed ("+" oriented), and f is a quotient of two analytic functions: $f = \frac{p}{q}$ (very often p, q are polynomials), use ③ or ④

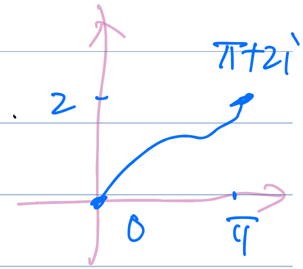
$\int_C g(z)\,dz$

③ v.s. ④

Tips: check how many bad pts of g are there inside C? 3 Scenarios

(I)	(II)	(III)
No bad pts inside C	1 bad pt z_0 inside C	≥ 2 bad pts inside C
↓	↓	↓
use ③	use ④	Next quarter

2 (a) Compute $\int_C \cos(\frac{z}{2})\, dz$, where C is a contour from $z=0$ to $\pi + 2i$.



A: Note: $\begin{cases} C \text{ is NOT closed;} \\ \left(2\sin(\frac{z}{2})\right)' = \cos\frac{z}{2} \text{ on } \mathbb{C} \end{cases}$

(Thus use anti-derivative method)

Choose $F = 2\sin(\frac{z}{2})$ $D = \mathbb{C} \Rightarrow C \subseteq D$

$$\Rightarrow \int_C \cos(\tfrac{z}{2})\,dz$$

$$= F(\pi + 2i) - F(0)$$

$$= 2\sin\left(\frac{\pi+2i}{2}\right) - 2\sin(0)$$

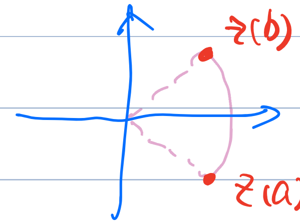
$$\underbrace{= \cdots}_{\text{Ex}} = e + \frac{1}{e}$$

Hint: $\sin w = \frac{e^{iw} - e^{-iw}}{2i}$

put in $w = \frac{\pi + 2i}{2} = \frac{\pi}{2} + i$

2(b). Compute $\int_C \frac{z+2}{\bar{z}}\,dz$, where C is the contour

$z(\theta) = e^{i\theta}$: $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$.



A: Use method of definition

$d\theta$

$$\int_C \frac{z+2}{\bar{z}}\,dz = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{z(\theta)+2}{\overline{z(\theta)}}\, z'(\theta)\,d\theta$$

$\boxed{\overline{e^{i\theta}} = e^{-i\theta}}$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{e^{i\theta}+2}{e^{-i\theta}}\left(ie^{i\theta}\right)d\theta$$

$\boxed{\text{Hint: } \frac{1}{e^{-i\theta}} = e^{i\theta}}$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(ie^{3i\theta} + 2ie^{2i\theta}\right)d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(ie^{3i\theta}\right)d\theta + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(2ie^{2i\theta}\right)d\theta$$

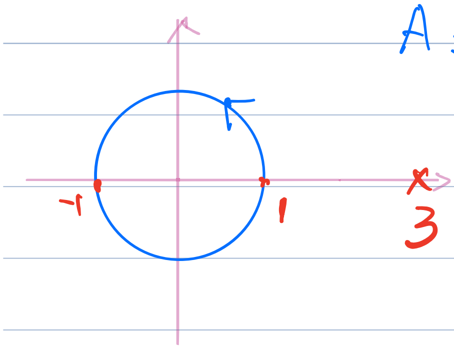
$$= \frac{1}{3}e^{3i\theta}\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + e^{2i\theta}\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \cdots = \left(2+\frac{\sqrt{2}}{3}\right)i$$

E.X

2 (c). Compute $\int_C \frac{z^2}{z-3}\,dz$, where C is $\underbrace{\text{the unit circle}}_{\text{radius}=1}$,

"+" oriented



A: let $g = \frac{z^2}{z-3}$

$\Rightarrow$ bad pt: $z-3=0$, $\Leftrightarrow z=3$

outside C

$\Rightarrow$ g: analytic $\begin{cases} \text{inside } C \\ \text{on } C \end{cases}$

By Cauchy - Goursat Thm

$$\int_C \frac{z^2}{z-3}\,dz = 0.$$

2 (d) Compute $\int_C \frac{1}{z^2+2z+2}\,dz$, where C is the closed unit circle, + oriented.

(Hint: $a,b,c \in \mathbb{C}$

$az^2+bz+c=0$, $a \neq 0$ $\Rightarrow$

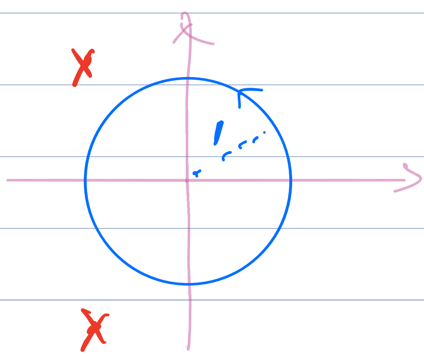
$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$)

A: Let $f = \frac{1}{z^2+2z+2}$

$a=1$ $b=2$ $c=2$

$\Rightarrow$ Bad pts: $z^2+2z+2=0$

$\Rightarrow$ Bad pts: $z = \frac{-2 \pm \sqrt{4-4\cdot 2}}{2}$ $\leftarrow \Delta = b^2-4ac = 4-4\cdot 2$



$= \frac{-2 \pm \sqrt{-4}}{2}$

$= \frac{-2 \pm 2\sqrt{-1}}{2}$

$= -1 \pm i$

Note: $\sqrt{-4} = \sqrt{4\cdot -1} = 2\sqrt{-1} = 2i$

Note $|-1 \pm i| = \sqrt{(-1)^2+(\pm 1)^2} = \sqrt{2} > 1$

$\Rightarrow$ Both bad pts are outside C

$\Rightarrow$ f : analytic $\begin{cases} \text{inside } C \\ \text{on } C \end{cases}$

By C.-G. Thm,

$$\Rightarrow \int_C f(z)\,dz = 0.$$

3(a) Compute $\int_C \frac{z^2}{z-3}\,dz$, where $\underbrace{C=\{|z|=4\}}_{\text{closed}}$.

"+" oriented

A: Note: Bad pt(s) of $\frac{z^2}{z-3}$: $z=3$

inside C (use C.I.F)

Let $f(z)=z^2$, $n=0$, $z_0=3$

$$\Rightarrow \int_C \frac{z^2}{z-3}\,dz = \int_C \frac{f(z)}{z-3}\,dz$$

$$= 2\pi i\, f(3) = 2\pi i \cdot 3^2 = 18\pi i$$

By C.I.F

3(b). Compute $\int_C \frac{z^2}{(z-3)^2}\,dz$, where $C = \{|z| = 4\}$.

closed

"+" oriented.

A: Again bad pts: $z = 3$, inside C

(Use C.I.F for D.)

Let $f(z) = z^2$ $(\Rightarrow f' = 2z)$, $z_0 = 3$, $n = 1$

$$\int_C \frac{z^2}{(z-3)^2}\,dz = \int_C \frac{f(z)}{(z-3)^2}\,dz$$

$$= \frac{2\pi i}{1!} f'(3) = 2\pi i \cdot 2 \cdot 3$$

$$= 12\pi i$$

by C.I.F for Derivatives.

3 (c) Compute $\int_C \frac{z^4}{(z-3)^5} dz$, where $C = \{|z| = 4\}$, "+" oriented.

closed

bad pts: $z=3$ inside C

A: let $f = z^4$, $z_0 = 3$, $n = 4$

$$\Rightarrow \int_C \frac{z^4}{(z-3)^5} dz = \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

by C.I.F for D.

$$= \frac{2\pi i}{4!} f^{(4)}(3)$$

E.x: $f = z^4 \Rightarrow f' = 4z^3$, $f'' = 12z^2$

$f''' = 24z$, $f^{(4)} = 24$

$$\Rightarrow \int_C \frac{z^4}{(z-3)^5} = \frac{2\pi i}{4!} 24 = 2\pi i$$

$$\boxed{4! = 24}$$

3(d). Compute $\int_C \frac{z}{(z-1)(z-3)}\,dz$, where

$C = \{|z-1|=1\}$, "+" oriented.

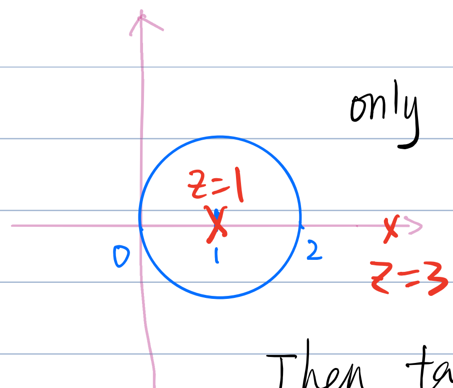
closed

A:

Note: bad pts of $\frac{z}{(z-1)(z-3)}$ are

$z=1$, $z=3$

only $z=1$ is inside C



Then take $\underline{z_0}=1$, $f=\frac{z}{z-3}$, $n=0$

$$\Rightarrow \int_C \frac{z}{(z-1)(z-3)}\,dz = \int_C \frac{f(z)}{z-1}\,dz$$

$$= 2\pi i\, \underline{f(1)} = 2\pi i \frac{1}{1-3} = -\pi i$$

by C.I.F

3(e). Compute $\int_C \frac{1}{4z^2+1}\,dz$, where $C=\{|z-\frac{1}{2}|=\frac{2}{3}\}$.

"+" oriented.

circle ⇒ closed

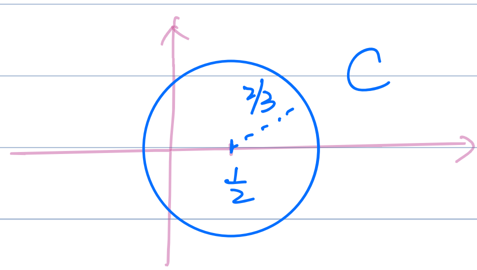
A: Step 1: Find bad pts of $\frac{1}{4z^2+1}$

Bad pts: $4z^2+1=0$

$\Rightarrow z^2+\frac{1}{4}=0 \Rightarrow z^2=-\frac{1}{4} \Rightarrow z=\pm\frac{1}{2}i$

(Hint: $4z^2+1=4(z^2+\frac{1}{4})=4(z+\frac{1}{2}i)(z-\frac{1}{2}i)$)

Step 2: Locate bad pts



$C: |z-\frac{1}{2}|=\frac{2}{3}$

v.s. $\pm\frac{1}{2}i$

distance from $\frac{1}{2}i$ to $\frac{1}{2}$ =

$$\left|\frac{1}{2}i-\frac{1}{2}\right|=\sqrt{\underbrace{(\tfrac{1}{2})^2}_{\frac{1}{4}}+\underbrace{(\tfrac{1}{2})^2}_{\frac{1}{4}}}=\sqrt{\tfrac{1}{2}}=\underbrace{\frac{\sqrt{2}}{2}}_{\frac{1.414\cdots}{2}>0.7}>\underbrace{\frac{2}{3}}_{0.666\cdots}$$

$\Rightarrow \frac{1}{2}i$ is outside C

Likewise, for $-\frac{1}{2}i$

$$\left| -\frac{1}{2}i - \frac{1}{2} \right| = \sqrt{(\tfrac{1}{2})^2 + (\tfrac{1}{2})^2} = \frac{\sqrt{2}}{2} > \frac{2}{3}$$

Hence both bad pts are outside C.

$\Rightarrow \frac{1}{4z^2+1}$: analytic $\begin{cases} \text{inside } C \\ \text{on } C \end{cases}$

By C.-G. Thm,

$$\Rightarrow \int_C \frac{1}{4z^2+1}\, dz = 0.$$

Q2: If v is a harmonic conjugate of u in $\mathbb{R}^2$, prove $-u$ is a harmonic conjugate of v in $\mathbb{R}^2$

Pf: There are 2 ways to prove it.

You can choose either way.

Way 1: (use defn)

(Recall: q is a harmonic conjugate of p

$\Longleftrightarrow \begin{cases} p_x = q_y \\ p_y = -q_x \end{cases}$ C.-R. eqns)

Since v is a harmonic conjugate of u.

$$\Longrightarrow \begin{cases} u_x = v_y & ① \\ u_y = -v_x & ② \end{cases}$$

Goal: verify $-u$ is a harmonic conjugate of v

By ②, $v_x = -u_y = (-u)_y$

By ① $v_y = u_x = -(-u)_x$

Hence by definition, $-u$ is a harmonic conjugate of v.

Way 2:

Fact: q is a harmonic conjugate of p

$\Longleftrightarrow$ $f \triangleq p + iq$ is analytic

$\Longleftrightarrow$ $\operatorname{Im} f$ is a har. conj. of $\operatorname{Re} f$

Since v is a harmonic conjugate of u.

$\Rightarrow$ $f = u + iv$ is analytic

$\Rightarrow$ $-if$ is analytic

$\boxed{-if = (-i)f}$

But $-if = -i(u+iv)$

$= -iu + v$

$= v + i(-u)$

Hence $-u$ is a harmonic conjugate of v.

☆☆☆☆☆

will upload a video to discuss Q1.

Q1: Use Cauchy-Riemann eqns to prove $f(z) = 2i\sin(\bar{z})$ is nowhere analytic.

A: step 1: Find u, v where $f = u + iv$.

$$\sin w = \frac{e^{iw} - e^{-iw}}{2i}$$

$$f(z) = 2i\sin(\bar{z})$$

$$= 2i\left(\frac{e^{i\bar{z}} - e^{-i\bar{z}}}{2i}\right)$$

$$= e^{i\bar{z}} - e^{-i\bar{z}}$$

$$= e \qquad - e$$

$$= e^{y}e^{ix} - e^{-y}e^{-ix}$$

$$= e^{y}(\cos x + i\sin x) - e^{-y}(\cos x - i\sin x)$$

$$= \underbrace{\cos x\,(e^{y}-e^{-y})}_{u} + i\,\underbrace{\sin x\,(e^{y}+e^{-y})}_{v}$$

Step 2. Solve C.-R. eqns

$$\begin{cases} u_x = v_y & ① \\ u_y = -v_x & ② \end{cases}$$

We get

$$① \Leftrightarrow (-\sin x)(e^{y}-e^{-y}) = \sin x\,(e^{y}-e^{-y})$$

$$\Leftrightarrow 2\sin x\,(e^{y}-e^{-y}) = 0$$

$$② \Leftrightarrow \cos x\,(e^{y}+e^{-y}) = -\cos x\,(e^{y}+e^{-y})$$

$$\Leftrightarrow 2\cos x\,(e^{y}+e^{-y}) = 0$$

Hence
$$\begin{cases} \sin x\,(e^{y}-e^{-y})=0 & (3) \\ \cos x\,(e^{y}+e^{-y})=0 & (4) \end{cases}$$

$(4) \Rightarrow$ either $\cos x = 0$ or $\underbrace{e^{y}+e^{-y}=0}_{\text{impossible as } e^{y}>0 \text{ and } e^{-y}>0}$

Hence we must have $\cos x = 0$

$\Rightarrow x = \frac{\pi}{2} + k\pi, \; k \in \mathbb{Z}$

$(3) \Leftrightarrow$ either $\sin x = 0$ or $e^{y} - e^{-y} = 0$

(But $\cos x = 0 \Rightarrow \sin x \neq 0$)

Hence we must have $e^{y} - e^{-y} = 0$

$\Leftrightarrow \; e^{y} = e^{-y}$

$\Leftrightarrow \; e^{2y} = 1 \quad \Leftrightarrow \; y = 0$

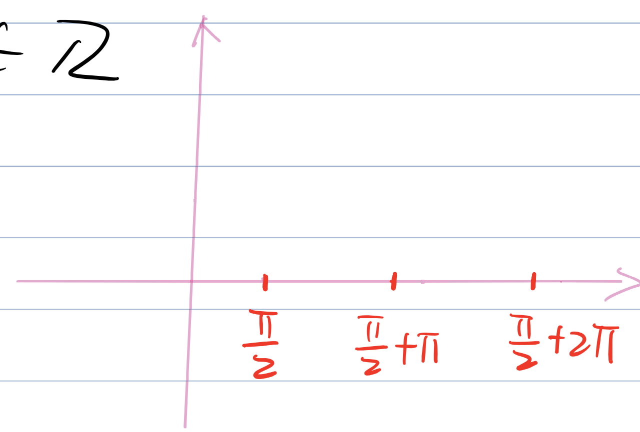
Summary: C.-R. eqns hold

$$\Longleftrightarrow \begin{cases} x = \frac{\pi}{2} + k\pi, \ k \in \mathbb{Z} \\ y = 0 \end{cases}$$

$$\Longleftrightarrow z = x + iy$$
$$= \frac{\pi}{2} + k\pi, \ k \in \mathbb{Z}$$

$\Longleftrightarrow$ $f(z)$ is $\mathbb{C}$-diff. precisely at

$\frac{\pi}{2} + k\pi, \ k \in \mathbb{Z}$



Hence there is no disk D s.t. f is $\mathbb{C}$-diff. in D

$\Rightarrow$ f is nowhere analytic.