Recall from last time:

How to compute a complex integral $\int_C f(z)\,dz$?

Method 1: (use definition)

- choose a parametrization of C: ← contour

$z(t), \quad a \le t \le b$

Compute using definition

$$\int_C f(z)\,dz = \int_a^b f'(z(t)) \cdot z'(t) \cdot dt$$

Today, we will a 2nd method:

to use anti-derivative

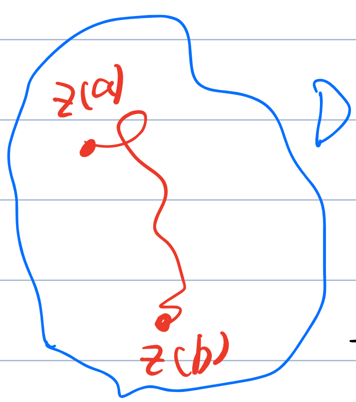
Antiderivative and path Independence

Lemma: let $f(z)$ be a continuous function on a domain $D \subseteq \mathbb{C}$.

Suppose $f(z)$ has an anti-derivative $F(z)$ on D

(meaning: F is an analytic function on D and $\underbrace{F'(z)}_{\text{complex derivative of } F} = f(z)$)

Assume the contour $C: z(t),\ a \le t \le b$ is entirely contained in D. Then



$$\int_C f(z)\,dz = \underbrace{F(z(b)) - F(z(a))}_{\text{depends only on } z(a),\, z(b)}$$

In particular, if C is closed (that is, $z(a) = z(b)$), then $\int_C f(z)\,dz = 0$.

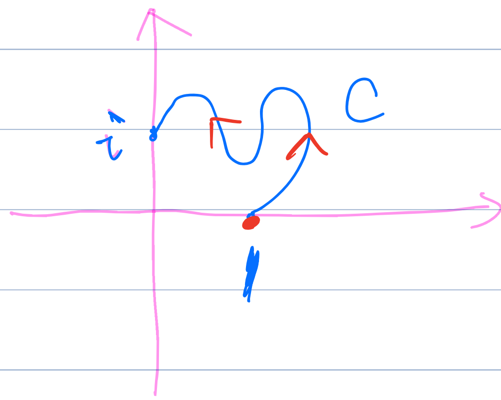
Tips on how to apply this Lemma:

To apply it, we need to find F and D, s.t.

① F analytic on D, $F' = f$

② D contains the entire contour C

E.g. let C be as below.



Hint:

$$\left(\frac{z^2}{2}\right)' = z$$

① $\int_C z\,dz$

Take $F = \frac{z^2}{2}$, $D = \mathbb{C}$

[F is analytic $\mathbb{C}$]

Note: ① C is entirely contained in D

② $F' = f$

By the lemma.

$$\int_C z\,dz = F(z(b)) - F(z(a))$$

$$= \frac{z^2}{2}\Bigg|_{z(a)=1}^{z(b)=i}$$

$$= \frac{i^2}{2} - \frac{1^2}{2} = -1$$

② $\int_C (z^2+i)\,dz$

Hint:

$$\left(\frac{z^3}{3}\right)' = z^2$$

$$(iz)' = i$$

we can take

$$F = \frac{z^3}{3} + iz$$

Take

$$F = \frac{z^3}{3} + iz$$

$$D = \mathbb{C}$$

By Lemma:

$$\int_C (z^2+i)\,dz = \left(\frac{z^3}{3} + iz\right)\Bigg|_1^i = -\frac{4}{3} - \frac{4}{3}i$$

Pf of lemma: (NOT required)

Pf:

<u>Claim</u>: We have the following
Chain Rule:

$$\frac{d}{dt}F(z(t)) = F'(z(t)) \cdot z'(t)$$

<u>pf of claim</u>:

$$\frac{F(z(t+\Delta t)) - F(z(t))}{\Delta t} = \frac{F(\overset{z}{\overbrace{z(t+\Delta t)}}) - F(\overset{z_0}{\overbrace{z(t)}})}{\underbrace{z(t+\Delta t)}_{z} - \underbrace{z(t)}_{z_0}} \cdot \underbrace{\frac{z(t+\Delta t) - z(t)}{\Delta t}}$$

let $\Delta t \to 0$ $\Big|$ $\qquad$ $\Big|_{\Delta t \to 0}$ $\qquad$ $\Big|_{\Delta t \to 0}$

$$\frac{d}{dt} F(z(t)) = F'(z(t)) \cdot z'(t)$$

Now

$$\int_C f(z)\,dz \overset{\text{by def}^n}{=} \int_a^b f(z(t)) \cdot z'(t)\,dt$$

$$= \int_a^b F'(z(t)) \cdot z'(t)\,dt$$

Recall $F' = f$

$$= \int_a^b \frac{d}{dt} F(z(t))\,dt$$

$$\underset{\text{F.T.C}}{=} F(z(t)) \Big|_a^b$$

$$= F(z(b)) - F(z(a))$$

Let's summarize some key things of the lemma:

Assume
- f, F defined on D;
- F = analytic on D, $F' = f$
- C entirely contained in D

$$\Rightarrow \int_C f(z)\,dz = F\Big|_{z(a)}^{z(b)}$$

Tips on how to use it:

Find F, D s.t

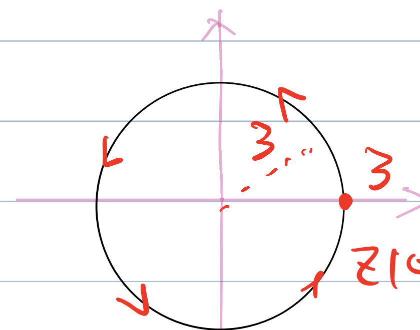
① $F' = f$ on D

② The contour C is entirely in D.

closed contour

E.g: ① Let C: $z(t) = 3e^{it}$, $0 \le t \le 2\pi$

compute $\int_C \frac{1}{z^2}\, dz$



$z(0) = z(2\pi) = 3$

Hint: Let $f = \frac{1}{z^2}$. Then

$$\left(-\frac{1}{z}\right)' = \frac{1}{z^2} \quad \left[\text{as } (z^{-1})' = -z^{-2}\right]$$

Hence $F = -\frac{1}{z}$ only bad pt: $z = 0$

How about D?

A: Take $F = -\frac{1}{z}$, $D = \mathbb{C} - \{0\}$

Then ① F: analytic on D ✓

$$F' = \frac{1}{z^2}$$

② C entirely in D ✓

Q: Can we take $D = \{1 < |z| < 4\}$?

or $D = \{|z| > 1\}$

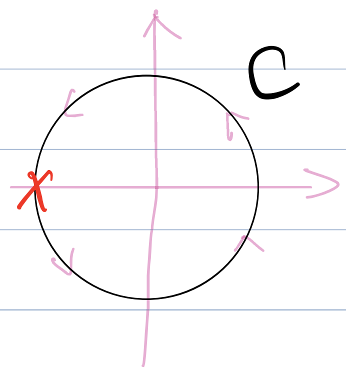
A: Yes!

By lemma,

$$\int_C \frac{1}{z^2} dz = F \Big|_{z(a)}^{z(b)} = 0 .$$

② Let C be as in ①:

$C: z(t) = 3e^{it}, \quad 0 \le t \le 2\pi$ ← Closed



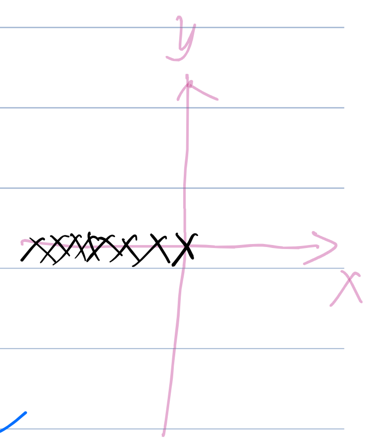
Compute $\int_C \frac{1}{z}\,dz$

Q: Recall $(\mathrm{Log}\, z)' = \frac{1}{z}$. Can we take $F = \mathrm{Log}\, z$?

A: Warning:

Log z is only analytic on

$\Omega = \{re^{i\theta} \mid r > 0, -\pi < \theta < \pi\}$



and $(\mathrm{Log}\, z)' = \frac{1}{z}$ on Ω

Then how about just take $D = \Omega$?

No!!! because Ω misses one pt of C, thus C is NOT entirely in Ω.

Indeed, we will not use the method of using anti-derivative.

Then how to compute it?

By definition!

$$\left(\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt \right)$$

$$\int_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{z(t)}\, z'(t)\,dt \qquad \boxed{\begin{aligned} &z(t) \\ &= 3e^{it} \end{aligned}}$$

$$= \int_0^{2\pi} \frac{1}{\cancel{3e^{it}}}\, \cancel{3}\, i\, \cancel{e^{it}}\,dt = \int_0^{2\pi} i\,dt$$

$$= 2\pi i$$

In general.

let $C: z(t) = 3e^{it}, \quad 0 \le t \le 2\pi$

- $\int_C \frac{1}{z^n} dz, \quad n \ge 2$

(Hint: $\left(-\frac{1}{n-1} \frac{1}{z^{n-1}}\right)' = \frac{1}{z^n}$)

Take $\begin{cases} F = -\frac{1}{n-1} \frac{1}{z^{n-1}} \\ D = \mathbb{C} - \{0\} \end{cases}$

$\Rightarrow \begin{cases} F' = f \text{ in } D \quad \checkmark \\ C \text{ entirely in } D \quad \checkmark \end{cases}$

$\Rightarrow \int_C \frac{1}{z^n} dz = 0. \quad n \ge 2$

- $n = 1, \quad \int_C \frac{1}{z} dz$

We use def^n to compute it!

$$\int_C \frac{1}{z}\, dz = 2\pi i$$