Recall:

- Let $w(t) : [a, b] \subseteq \mathbb{R} \to \mathbb{C}$

  write $w(t) = u(t) + i\,v(t)$

  then $\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i \int_a^b v(t)\,dt$

- Consequence of F.T.C.

  If $F'(t) = w(t)$,

  $\int_a^b w(t)\,dt = F(b) - F(a)$

$\int_a^b w(t)\,dt \qquad w: [a,b] \subseteq \mathbb{R} \to \mathbb{C}$

we will finally define the integral of a complex function $f(z)$. ($f: \mathbb{C} \to \mathbb{C}$). we will define the integral of $f(z)$ over a contour (also called a curve)

Q: What is a Contour (Curve) ?

② Contour (Curves)

Def^n: Let $x(t)$, $y(t)$ be continuous real functions on $[a, b]$,

Then $z(t) \triangleq x(t) + i\,y(t)$ is called an arc or a curve.

$$z : t \in [a, b] \to \mathbb{C}$$

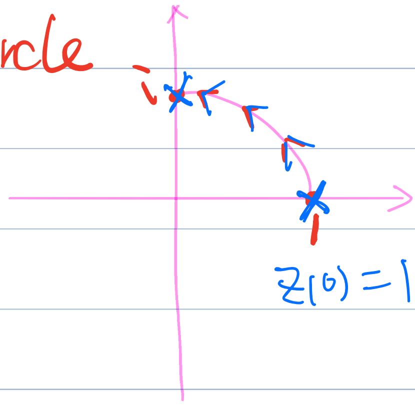
$0 \to t \to \frac{\pi}{2}$

E.g.: ① $z(t) = \underbrace{\overset{x(t)}{\cos t} + i\,\overset{y(t)}{\sin t}}_{e^{it}}$, $t \in [0, \frac{\pi}{2}]$

e^{it}, $Arg(z(t)) = t \in [0, \frac{\pi}{2}]$

$\Rightarrow |e^{it}| = 1$

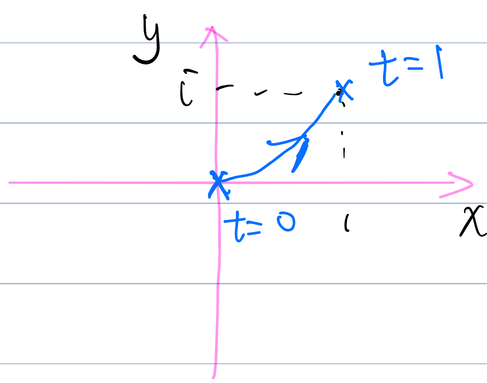
Note:
z(t) is on
the unit circle



when $t = 0$, $z(t) = e^{i0} = 1$

when $t = \frac{\pi}{2}$, $z(t) = e^{i\frac{\pi}{2}} = i$

② $z(t) = \overset{x(t)}{t} + i\,\overset{y(t)}{t^2}$, $t \in [0, 1]$



Note: $x = t$
$y = t^2$

$\Rightarrow y = x^2$

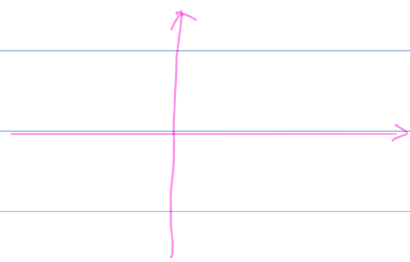
$\Rightarrow$ z(t) is always on $\{y = x^2\}$

$t = 0 \Rightarrow z(t) = 0 + i0^2 = 0$

$t = 1 \Rightarrow z(t) = 1 + i$

③ $z(t) = t$ $\quad t \in [-1, 1]$.

Ex

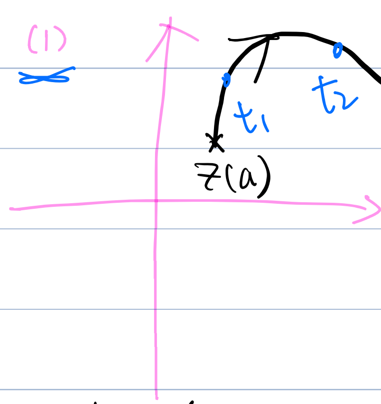


Defn: An arc/curve $z(t)$ is called a simple arc/curve

$\iff$ It has no self-intersection

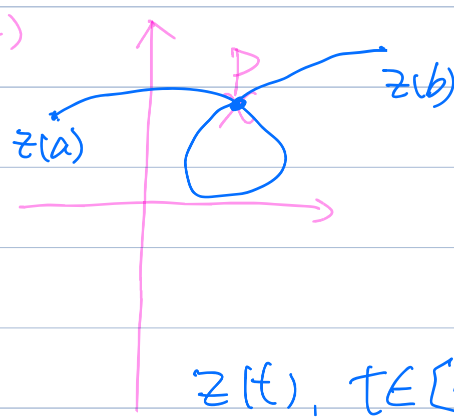
(meaning, $z(t_1) \neq z(t_2)$ for all $t_1 \neq t_2$)

E.g: (1)



$z(t)$, $t \in [a, b]$

(2)

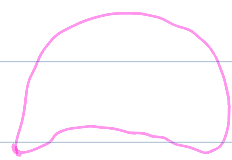


$z(t)$, $t \in [a, b]$

$t \in [a, b]$

Def^n: $z(t)$ is a "simple closed" curve or "Jordan curve" $\Longleftrightarrow$

$z(t)$ does NOT have self-intersection except $z(a) = z(b)$



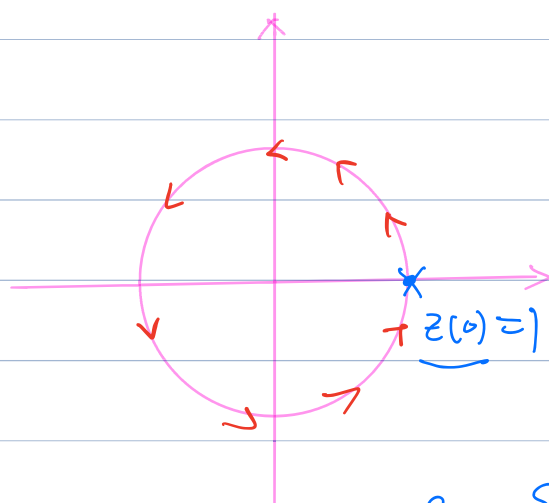
$z(a) = z(b)$

E.g

$$z(t) = \cos t + i \sin t = e^{it}, \quad t \in [0, 2\pi]$$

$t = 0, \Rightarrow z(0) = e^{i0} = 1$

$t = 2\pi \Rightarrow z(2\pi) = e^{i2\pi} = 1$



$z(0) = 1 = z(2\pi)$

a simply closed curve.

Def^n: $z(t)$ is called a smooth arc (or a smooth curve) $\iff$ $z'(t)$ is continuous and nowhere zero

---

consider the above e.g

$$z(t) = \cos t + i \sin t$$

$$\Rightarrow z'(t) = -\sin t + i \cos t$$

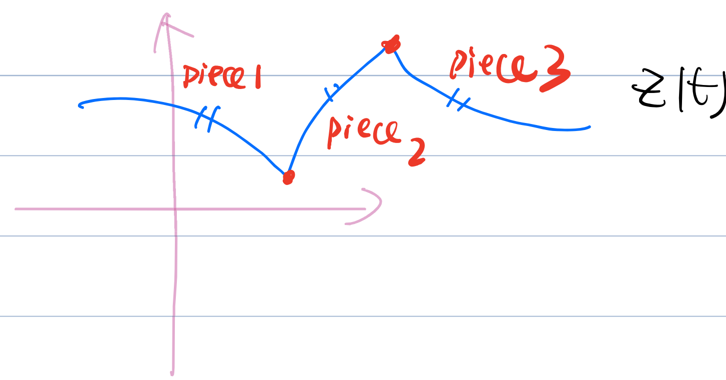
(Note: $\sin^2 t + \cos^2 t = 1 \Rightarrow$ $\sin t$, $\cos t$ cannot be both 0
$\Rightarrow z'(t) \neq 0$.)

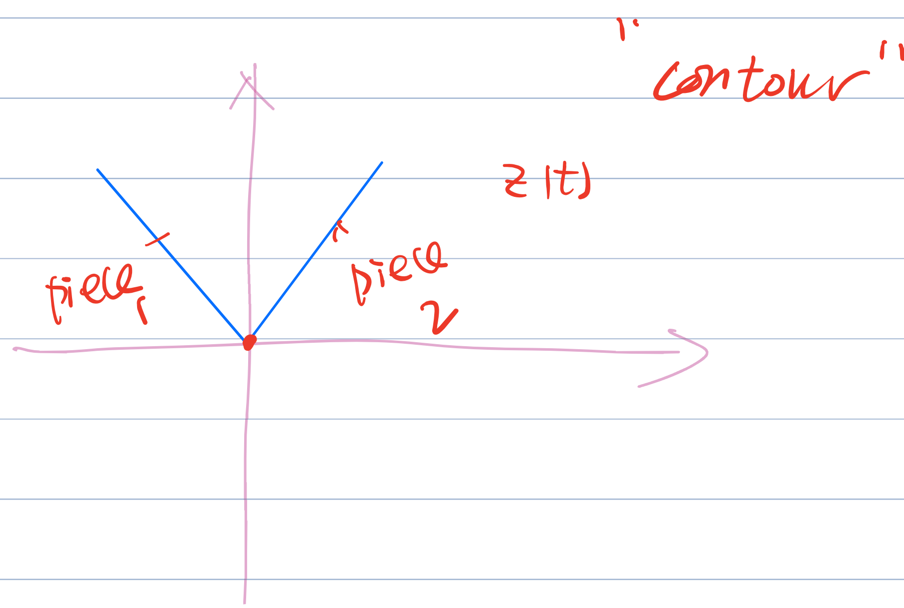
Thus $z(t)$ is a smooth curve.

Def^n :

① $z(t)$, $a \leq t \leq b$ is called a contour $\Longleftrightarrow$ a piecewise smooth arc/curve

E.g





② closed contour $\Longleftrightarrow$ $\begin{cases} \text{contour} \\ \text{closed: } z(a)=z(b) \end{cases}$

③ simple closed contour $\Longleftrightarrow$

$\begin{cases} \text{closed: } z(a)=z(b) \\ \text{contour: piecewise smooth} \\ \text{simple closed:} \end{cases}$

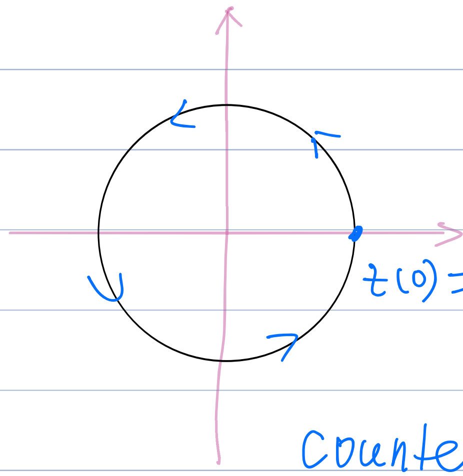
no self-intersection

except $z(a)=z(b)$

$0 \to t \to 2\pi$

E.g ① $z(t) = e^{it}, \quad 0 \le t \le 2\pi$

$\operatorname{Arg} z(t) = t$



This is a simple closed contour

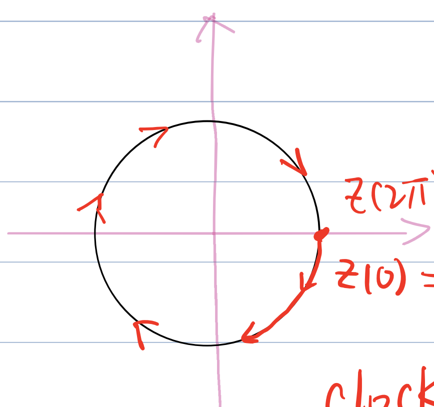
$z(0) = z(2\pi)$

counterclockwise

$0 \to t \to 2\pi$

② $z(t) = e^{-it} \quad 0 \le t \le 2\pi$

$\arg z(t) = -t + 2n\pi$



$z(2\pi) = 1$

$z(0) = 1$

clockwise.

Warning: we regard the above ①, ② as different contours, as they have distinct orientations

One more definition:

(1) The length of a smooth curve, say $z(t) = x(t) + i\,y(t)$, $a \le t \le b$ is defined as

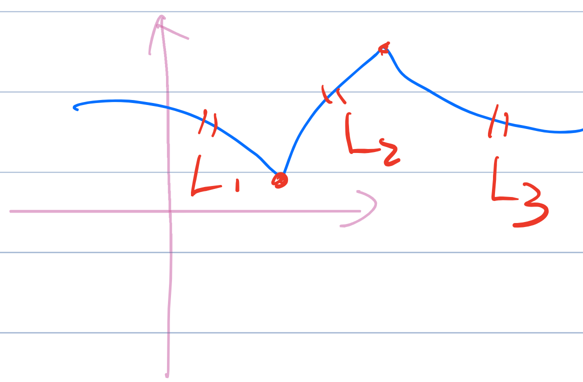
$$L \triangleq \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_a^b |z'(t)|\, dt$$

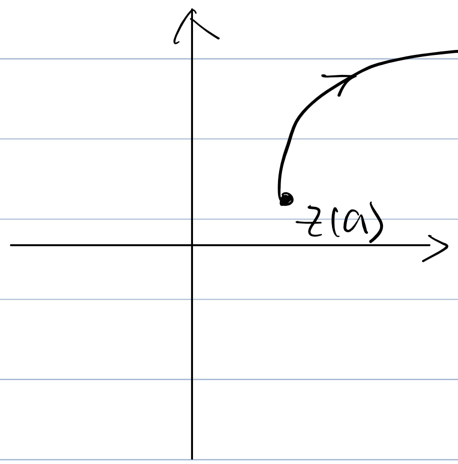
(Note: $z'(t) = x'(t) + i y'(t)$)

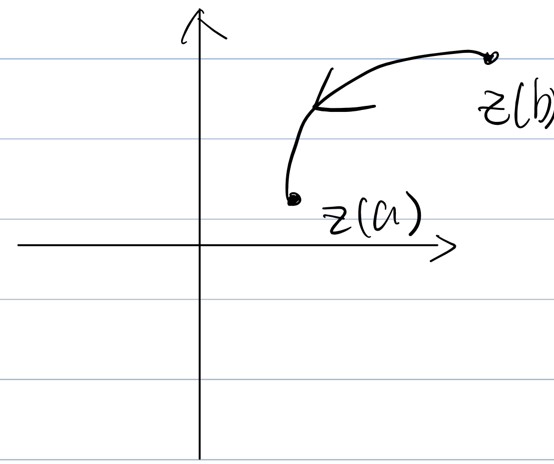
② length of a contour defined by piecewise smooth curve

= the sum of the lengths of each smooth piece



Arc with reversing orientation:





Let C: $z(t)$

$a \le t \le b$

Let $\tilde{C}$:

$w(t) = z(-t)$

$-b \le t \le -a$

$z(b) \longrightarrow z(a)$

Remark: (we will write $\tilde{C}$ as $-C$)

① C and $-C$ are considered as different contours as they have different orientation

② $\text{length}(C) = \text{length}(-C)$

Next we define complex integral (a.k.a. contour integral)

Def^n: Let $C: z(t),\ a \le t \le b$ be a contour

Let $f(z)$ be a function defined on C and continuous on C

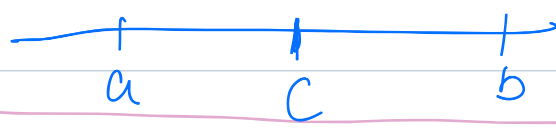
we define the contour integral

$$\int_C f(z)\,dz \triangleq \int_a^b f(z(t))\, z'(t)\, dt$$

Remark:

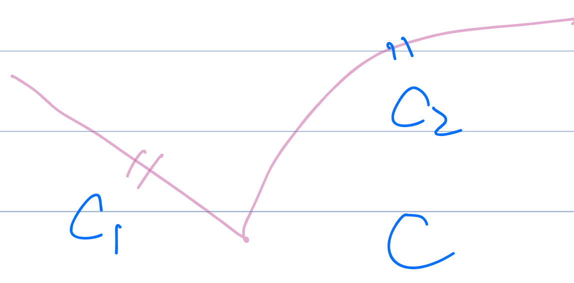
Recall: In Calculus,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



① If $C = C_1 + C_2$, then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$



② we can understand the definition as follows: $C = z(t),\ a \le t \le b$

$$dz = \frac{dz}{dt}\,dt = z'(t)\,dt$$

$$\Rightarrow \int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt$$

$$z \in C \Rightarrow z = z(t),\quad a \le t \le b$$

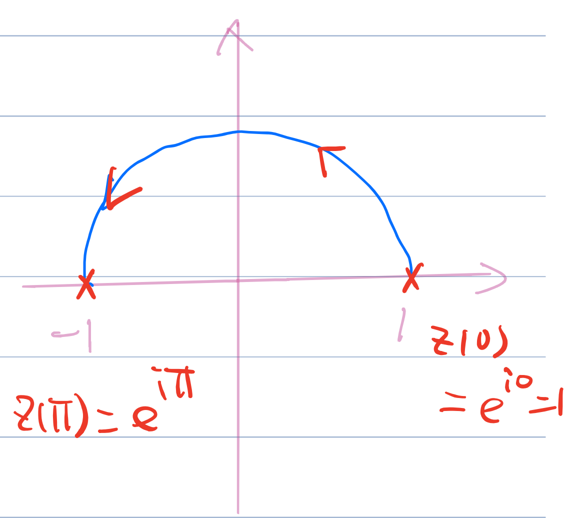
E.g: ① $C:\ z(t) = e^{it};\ 0 \le t \le \pi$

Compute $\int_C z\,dz$

B

semi-circle

counterclockwise



A: By defn,

$$\int_C z\,dz = \int_0^{\pi} z(t)\, z'(t)\, dt$$

Here $z(t) = e^{it}$

$$\Rightarrow \quad z'(t) = e^{it} \cdot i$$

Last time:

$$\frac{d}{dt} e^{z_0 t} = e^{z_0 t} \cdot z_0$$

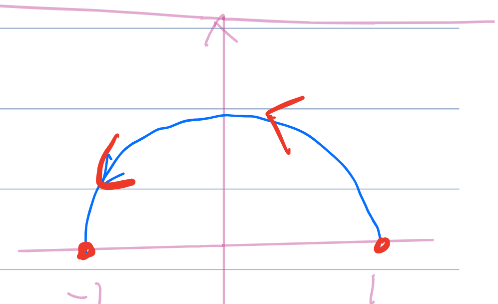
Therefore,

$$\int_C z\,dz = \int_0^{\pi} e^{it}\, i\, e^{it} dt = \int_0^{\pi} i\, e^{2it} dt$$

$$= \left[\frac{1}{2} e^{2it}\right]\Big|_0^{\pi} = \frac{1}{2}\left(e^{2i\pi} - e^{2i\cdot 0}\right) = 0$$

② C: $z(t) = e^{2it}$, $0 \le t \le \frac{\pi}{2}$

Compute $\int_C z\,dz$



$z(\frac{\pi}{2}) = e^{i\pi} = -1$

$z(0) = 1$

$z(t) = e^{2it}$

$\Rightarrow z'(t) = 2i\, e^{2it}$

semi-circle counterclockwise

A:

$$\int_C z\, dz = \int_a^b z(t)\, z'(t)\, dt$$

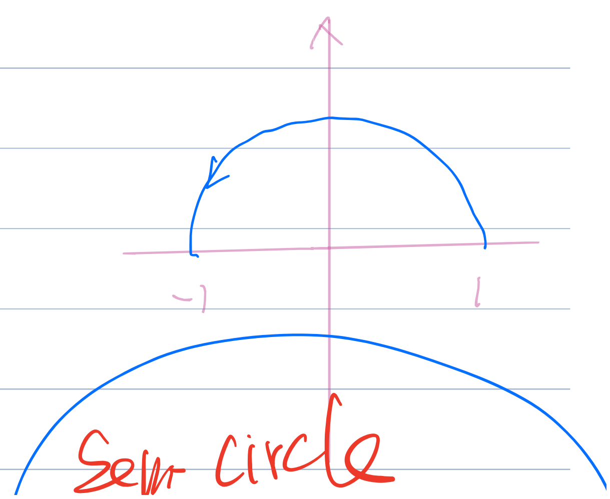
$$= \int_0^{\pi/2} e^{2it}\, 2i\, e^{2it}\, dt$$

$$= \int_0^{\pi/2} 2i\, e^{4it}\, dt$$

$$= \left[\frac{1}{2} e^{4it}\right]\Big|_0^{\pi/2} = \frac{1}{2}\left[e^{4i\frac{\pi}{2}} - e^{4i\cdot 0}\right] = 0$$

③ $C: z(t) = e^{3it} \quad 0 \le t \le \pi/3$

Compute $\int_C z\, dz$



counterclock

A: E.x

compute and check

the answer $= 0$.

Remark: In general, the integral

$\int_C f(z)\,dz$ does NOT depend on the parametrization of the contour C (need to keep the orientation)

More precisely,

If $\begin{cases} z_1(t), & a_1 \le t \le b_1 \\ z_2(t), & a_2 \le t \le b_2 \end{cases}$ parametrize

the same contour (with the same orientation)

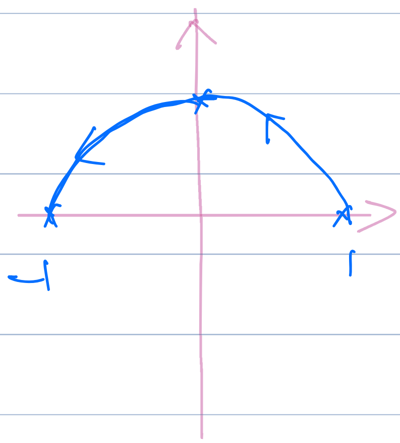
then the computation of $\int_C f(z)\,dz$ using

$z_1(t), z_2(t)$ give the same answer.

E.g ① Compute $\int_C \bar{z}\, dz$

C: is the semi-circle in the graph.

counter-clockwize oriented



Take any parametrize of C as you like:

$z(t) = e^{it} \quad 0 \le t \le \pi$