Pratice Problems for Midterm II

Note: No books, notes, cheat sheets, calculator or any electronic devices are allowed during Midterm II exam.

Tips:

- You are required memorize(not limited to):

  Cauchy-Riemann equations in $xy-$coordinates, and the formula $f' = u_x + iv_x$;

  Definition of harmonic functions, Laplace equation, harmonic conjugate;

  Definition of exponential functions, logarithmic functions, power functions, sine and cosine functions

- You don't need to memorize the Cauchy-Riemann equations in polar form. If it will be needed in the exam, we will provide it.

1. Write $z = x + iy$ with $x, y \in \mathbb{R}$. Let $f(z) = x^2 + iy^2$. Find all points where f is complex differentiable and find the value of f' at these points. Then find where f is analytic. Show your work.

2. Let $u(x, y) = 2xy - x$.

   (a) Prove u is harmonic on $\mathbb{R}^2$.

   (b) Find all harmonic conjugates v of u on $\mathbb{R}^2$..

   (c) Let v be your answer in (b). Find $f(z)$ such that $f = u + iv$. Express the function f in terms of z. Show your work.

3. State the definitions of $\mathrm{Log} z$ and $\log z$. Then find $\log e$ and $\mathrm{Log}(-ei)$. Show your work.

4. Does $\mathrm{Log}(i^3)$ equal to $3\mathrm{Log} i$? Justify your answer.

5. Let $c \in \mathbb{C}$. State the definitions of z^c and P.V.z^c. Then find $\frac{1}{i^{2i}}$. Show your work.

6. State the definitions of $\sin z$ and $\cos z$. Then use the definitions to prove $\cos z = \sin(z + \frac{\pi}{2})$.

Q1: Let $f(z) = x^2 + iy^2$

Check where f is $\mathbb{C}$-diff'ble.

compute $f'(z)$ at such points.

find where f is analytic.

Step 1: find u, v. compute u_x, u_y, v_x, v_y

Note: $f = \underbrace{x^2}_{u} + i\underbrace{y^2}_{v}$

compute $\begin{cases} u_x = 2x \\ u_y = 0 \end{cases}$; $\begin{cases} v_x = 0 \\ v_y = 2y \end{cases}$

Step 2: Check whether (at what points) the C.-R. eqns hold.

C.R.-eqns. $\begin{cases} u_x = v_y & ① \\ u_y = -v_x & ② \end{cases}$

Check: ①: $2x = 2y$?

$\Rightarrow$ ① holds iff $x = y$.

②: $0 = -0$?

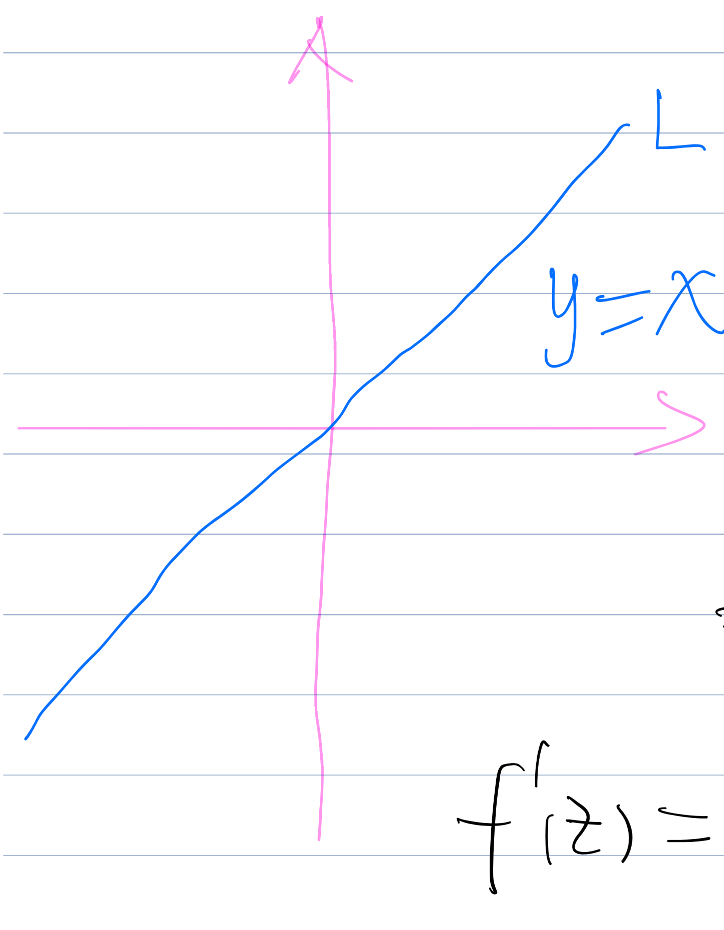
$\Rightarrow$ ② holds always

Hence, the two C.R. eqns holds precisely at $x = y$

As a result, f is

$\mathbb{C}$-diff^{ble} precisely

at $z = x + ix$, $x \in \mathbb{R}$



$f(z) = x^2 + iy^2$

At such pts

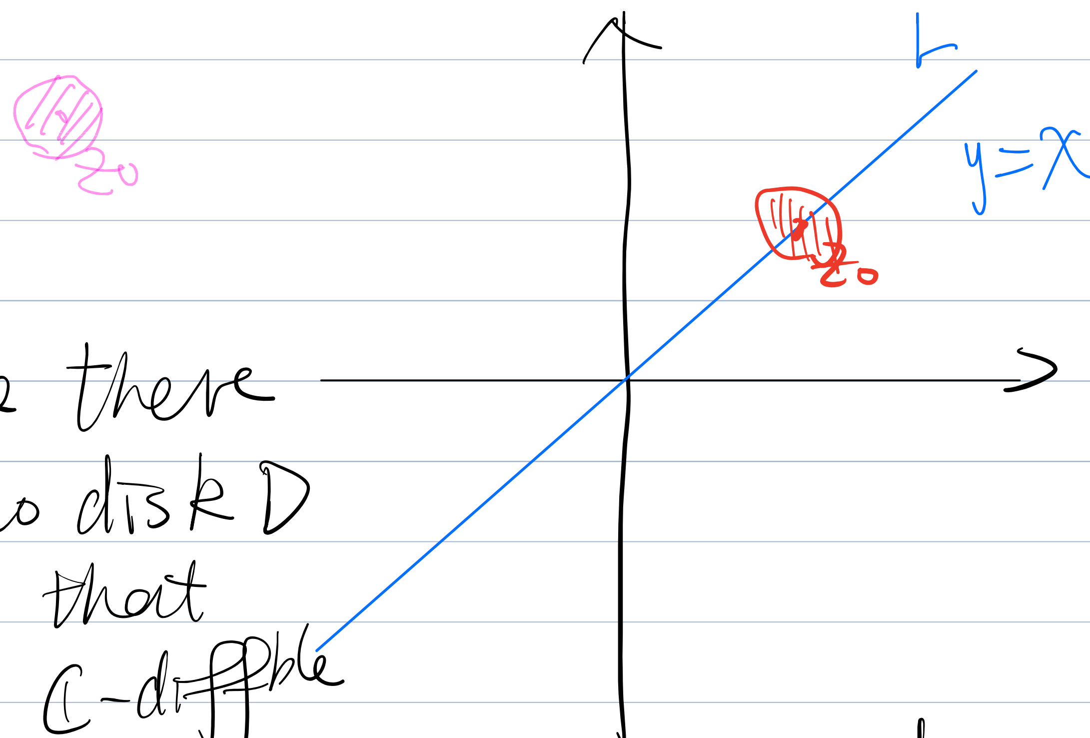
$z = x + ix$,

$$f'(z) = u_x + iv_x$$

$$= 2x + i \cdot 0 = 2x$$

Step 3. Find where f is analytic?

(Recall f is analytic at z_0
$\Leftrightarrow$ f is $\mathbb{C}$-diff'ble in a nbhd of z_0: $\{|z-z_0|<\varepsilon\}$)



Since there is no disk D such that f is $\mathbb{C}$-diff'ble in D. $\Rightarrow$ f is nowhere analytic.

Q2: Let $u = 2xy - x$

(a) prove u is harmonic in $\mathbb{R}^2$

Pf: (Recall u is harmonic

$\Longleftrightarrow u_{xx} + u_{yy} = 0$)

(a) Compute $u_x = 2y - 1$, $u_{xx} = 0$

$u_y = 2x$, $u_{yy} = 0$

$\Longrightarrow u_{xx} + u_{yy} = 0$

(b) Find all harmonic conjugates v of u in $\mathbb{R}^2$.

A:

Step 1: Find V_x, V_y

(Recall v is a harmonic conjugate of u

$\Longleftrightarrow \begin{cases} u_x = v_y & (1) \\ u = -v_x & (2) \end{cases}$)

By 2 $\Rightarrow$

$$v_x = -u_y = -2x \quad ③ \checkmark$$

By ① $\Rightarrow$ $v_y = u_x = 2y - 1$ ④ $\checkmark$

Step 2. Integrate ④ w.r.t y

(regard x as a constant)

$$v = \int v_y \, dy$$

$$= \int (2y-1) \, dy$$

$$= y^2 - y + \phi(x)$$

"constant"

Step 3. Bring '$v = y^2 - y + \phi(x)$'

to ③: $v_x = -2x$

$$\Rightarrow \quad v_x = \frac{\partial}{\partial x}\left(y^2 - y + \phi(x)\right) = -2x$$

$$\Rightarrow \quad \phi'(x) = -2x$$

$$\Rightarrow \quad \phi = \int -2x \, dx$$

$$= -x^2 + C \leftarrow \text{constant}$$

Hence

$$v = y^2 - y + \phi$$

$$= y^2 - y - x^2 + C, \quad C \in \mathbb{R}$$

(C) Find $f(z)$ s.t $f(z) = u + iV$

Express $f(z)$ in term of z.

(v is a harmonic conjugate of u

$\Longleftrightarrow f = u + iv$ is analytic.)

$$f = u + iV$$

$$= 2xy - x + i(y^2 - y - x^2 + c)$$

$$= \underbrace{iC}_{\text{constant}} + \underbrace{-x - iy}_{\text{linear}} + \underbrace{2xy + iy^2 - ix^2}_{\text{quadratic}}$$

$$= iC + \underbrace{-(x+iy)} - iz^2$$

$-z$

$= ic - z - iz^2$

$\Rightarrow$

$f = ic - z - iz^2$

Hint:

$z^2 = (x+iy)^2$

$= x^2 + 2ixy - y^2$

$\Rightarrow$

$-iz^2$

$= -ix^2 + 2xy$

$+ iy^2$

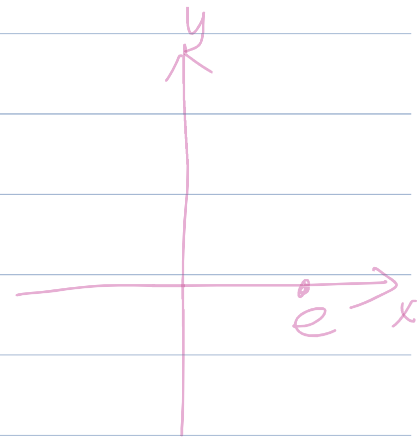
Q3: State the definition of Log z, log z

compute log e, Log(−ei)

A: $\log z = \ln|z| + i\arg z \; ; \; z \neq 0$

$\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z$

Compute $\log e$, $\mathrm{Log}(-ei)$



$$\log e = \ln|e| + i\arg e$$
$$= \ln e + i\arg e$$
$$= 1 + i(0+2n\pi)$$
$$= 1 + i2n\pi,\quad n\in\mathbb{Z}$$

E.x: $\mathrm{Log}(-ei) = 1-\frac{\pi}{2}i$

Recall: $x\in\mathbb{R}$

in Calculus, $\ln x^k = k\ln x$, $x>0$

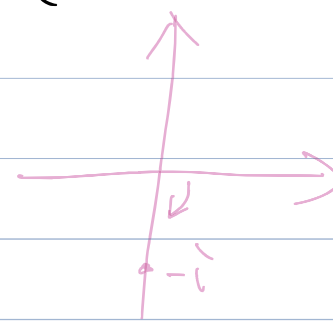
$k\geq 1$, $k\in\mathbb{Z}$

In complex analysis, Can we expect the same ?

Q4: Does $\mathrm{Log}(i^3) = 3\,\mathrm{Log}\, i$? ✗

A: No! why?

$$\text{LHS} = \mathrm{Log}(i^3) = \mathrm{Log}(-i)$$
$$= \ln|-i| + i\,\mathrm{Arg}(-i)$$
$$= \overset{0}{\cancel{\ln 1}} + i\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} i$$



$$\text{RHS} = 3\,\mathrm{Log}\, i$$
$$= 3\left(\overset{0}{\cancel{\ln|i|}} + i\,\overset{\frac{\pi}{2}}{\cancel{\mathrm{Arg}\, i}}\right)$$
$$= i\,\frac{3\pi}{2}$$

Hence, LHS $\neq$ RHS

Q5. State the definition of z^c and P.V. z^c.

find $\dfrac{1}{i^{2i}}$.

A: $$z^c = e^{c \log z}$$

$$\text{P.V. } z^c = e^{c \operatorname{Log} z}$$

To compute $\dfrac{1}{i^{2i}}$, we first compute

$$\begin{aligned} i^{2i} &= e^{2i \log i} \\ &= e^{2i(\cancelto{0}{\ln|i|} + i \cancelto{\pi/2 + 2n\pi}{\arg i})} \\ &= e^{2i(i\pi/2 + i2n\pi)} \\ &= e^{-\pi - 4n\pi} \end{aligned}$$

$$\Rightarrow \frac{1}{i^{2i}} = e^{\pi + 4n\pi}, \quad n \in \mathbb{Z}$$

Q6: State the definition of $\sin z$, $\cos z$.

prove $\cos z = \sin(z + \frac{\pi}{2})$

A:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

To prove "$\cos z = \sin(z + \frac{\pi}{2})$",

we note

$$RHS = \sin(z + \frac{\pi}{2})$$

$$= \frac{e^{i(z+\frac{\pi}{2})} - e^{-i(z+\frac{\pi}{2})}}{2i}$$

Hint:

$e^{i\frac{\pi}{2}} = i$

$e^{-i\pi/2} = -i$

$$= \frac{e^{iz}(\cancel{i}) - e^{-iz}(-\cancel{i})}{2\cancel{i}}$$

$$= \frac{e^{iz} + e^{-iz}}{2} = \cos z = LHS$$

In the remaining time, we continue to discuss some new material.

Recall in last lecture:

Let $w(t)$ be a complex-valued of a real variable:

$$w(t): [a,b] \to \mathbb{C}$$

- Integral: write $w(t) = u(t) + iV(t)$

$$\int_a^b w(t)\,dt \triangleq \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

- If $F(t) = U(t) + iV(t)$, $w(t) = u(t) + iv(t)$

we say F is an antiderivative of w (denoted by $F' = w$) $\iff$

$$U' = u, \quad V' = v$$

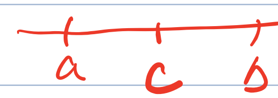
Consequence of F. T. C.

If $F'(t) = w(t)$,

$$\int_a^b w(t)\,dt = F(b) - F(a)$$

Useful formulas:

- $$\left| \int_a^b w(t)\,dt \right| \leq \int_a^b |w(t)|\,dt$$
- $$\int_a^b w(t)\,dt = \int_a^c w(t)\,dt + \int_c^b w(t)\,dt$$



- $$\int_a^b w(t)\,dt = - \int_b^a w(t)\,dt$$

$a \to b$ $\qquad$ $b \to a$

$$\int_a^b w(t)\,dt \qquad w: [a,b] \subseteq \mathbb{R} \to \mathbb{C}$$

we will finally define the integral of a complex function $f(z)$. ($f: \mathbb{C} \to \mathbb{C}$). we will define the integral of $f(z)$ over a contour (also called a curve)

Q: What is a Contour (Curve) ?

② Contour (Curves)

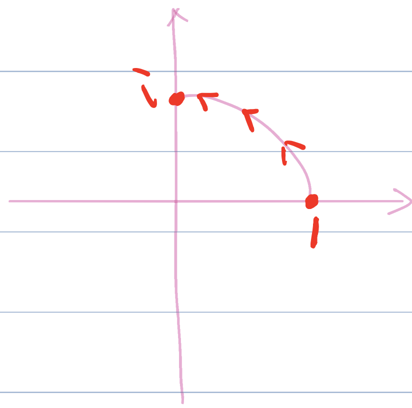
Def^n: Let $x(t)$, $y(t)$ be continuous real functions on $[a, b]$,

Then $z(t) \triangleq x(t) + i y(t)$ is called an arc or a curve.

E.g.: ① $z(t) = \underbrace{\cos t + i\sin t}_{e^{it}}$, $t \in [0, \frac{\pi}{2}]$

e^{it}, $Arg(z(t)) = t \in [0, \frac{\pi}{2}]$

$\Rightarrow |e^{it}| = 1$

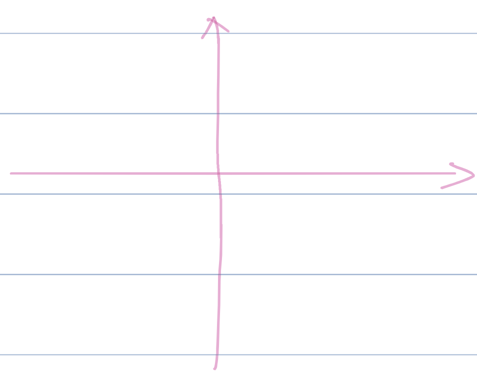


when $t=0$, $z(t) = e^{i0} = 1$

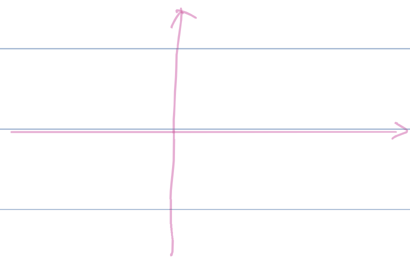
$\downarrow$

when $t = \frac{\pi}{2}$, $z(t) = e^{i\frac{\pi}{2}} = i$

② $z(t) =$



③ $z(t) = t \qquad t \in [-1, 1]$.

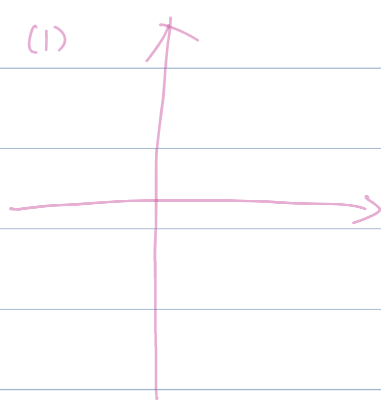


Def^n: An arc/curve $z(t)$ is called a simple arc/curve

$\Longleftrightarrow$

E.g.:

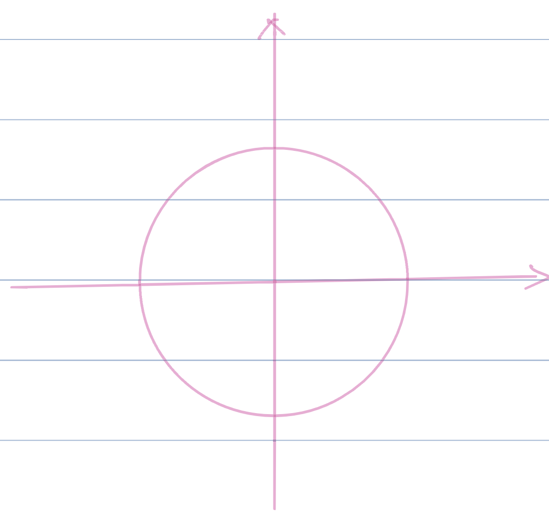
(1)



(2)

Def^n: $z(t)$ is a "simple closed" curve or "Jordan curve" $\Longleftrightarrow$

E.g $z(t) =$



Def^n: $z(t)$ is called a smooth arc (or a smooth curve) $\iff$