Recall:

Important conventions we will use in the future:

$\ln x$: real log function

$\log z$: complex log function

$\log z = \ln|z| + i \arg z$

$= \ln|z| + i \operatorname{Arg} z + i 2n\pi, \ n \in \mathbb{Z}$

$\operatorname{Log} z$: principal value of complex log function

$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$

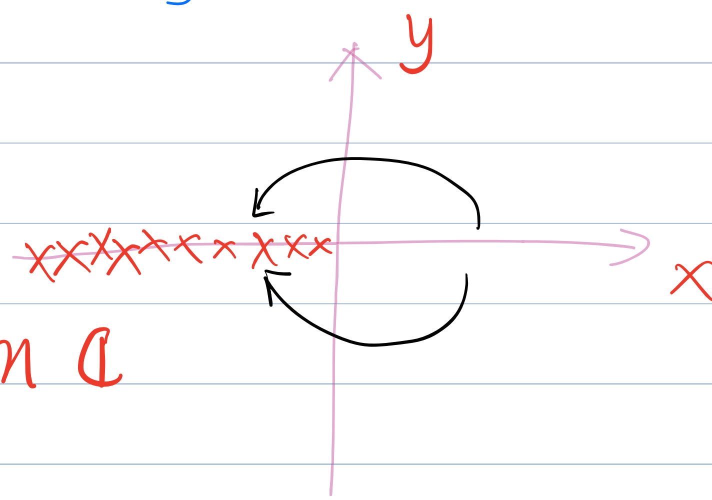
$\operatorname{Arg} z \in (-\pi, \pi]$

Warning: $e^z \longleftrightarrow \operatorname{Log} z$

$\operatorname{Log} z$ is NOT analytic on $\mathbb{C}$

why?

$\operatorname{Log} z$ is NOT even continuous on $\mathbb{C}$



To make $\operatorname{Log} z$ analytic, we delete 0 and delete the negative x-axis from $\mathbb{C}$:

$$D = \{ z = re^{i\theta} \mid r > 0, \ -\pi < \theta < \pi \}$$

$\Rightarrow$ $\operatorname{Log} z$ is analytic on D

$$\frac{d}{dz}(\operatorname{Log} z) = \frac{1}{z},$$

$e^z \longleftrightarrow \log z \,/\, \mathrm{Log}\, z$

Remark:

- For all $z \neq 0$ in $\mathbb{C}$, we have

$$e^{\log z} = z \quad \text{and} \quad e^{\mathrm{Log}\, z} = \underline{z}$$

Why?
$$e^{\log z} = e^{\ln|z| + i\,\mathrm{Arg}\, z + i2n\pi}, n \in \mathbb{Z}$$
$$= e^{\ln|z| + i\,\mathrm{Arg}\, z}\, e^{i2n\pi}$$
$$= e^z$$

$$\boxed{e^{i2n\pi} = 1 \quad n \in \mathbb{Z}}$$

- How about $\log(e^z)$? Does $\log(e^z) = z$?

Warning: $\log(e^z) \neq z$ in general

Indeed, $\log(e^z) = z + i2n\pi$.

• How about $\text{Log}(e^z)$: Does $\text{Log}(e^z) = z$?

Warning: In general, $\text{Log}(e^z) \neq z$. $(*)$

Why? See the following examples

E.g 1: let $z = 2\pi i$

$$e^z = e^{2\pi i} = 1$$

$$\text{LHS of } (*) = \text{Log}(e^z)$$

$$= \text{Log}\,1$$

$$= \ln 1 + i\,\text{Arg}\,1$$

$$= 0 + i \cdot 0 = 0$$

$$\text{RHS of } (*) = z = 2\pi i$$

Thus

$$\text{Log}(e^z) \neq z$$

E.g 2: let $z = 0$

$$e^z = e^0 = 1$$

$$\text{LHS of } (*) = \text{Log}(e^z)$$

$$= \text{Log}\,1 = 0$$

$$\text{RHS of } (*) = z = 0$$

Remark:

Recall in Calculus:

$$\ln(x_1 x_2) = \ln x_1 + \ln x_2 \quad \text{if } x_1, x_2 > 0.$$

Q: How about $\log z$ and $\operatorname{Log} z$?

A:

- $\log(z_1 z_2) = \log z_1 + \log z_2$ if $z_1, z_2 \neq 0$

$$= \ln|z_1| + \ln|z_2| + i(\operatorname{Arg} z_1 + \operatorname{Arg} z_2) + i2n\pi, \quad n \in \mathbb{Z}$$

- However,

$$\operatorname{Log}(z_1 z_2) \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$$

in general.

Why?. E.g: Take $z_1 = z_2 = -1$

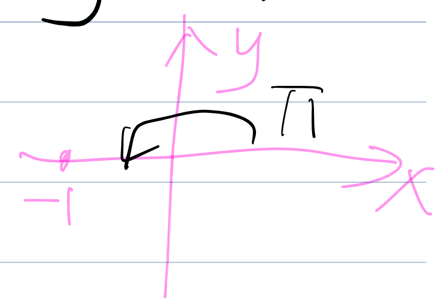
$$\operatorname{Log}(z_1 z_2) = \operatorname{Log}(1) = 0$$

$$\operatorname{Log} z_1 + \operatorname{Log} z_2 = \operatorname{Log}(-1) + \operatorname{Log}(-1)$$

$$= 2 \operatorname{Log}(-1)$$

$$= 2\left(\cancelto{0}{\ln|-1|} + i\,\overbrace{\operatorname{Arg}(-1)}^{\pi}\right)$$

$$= 2(i\pi) = 2\pi i$$



Power functions

In Calculus, we have power functions,

e.g., x^2, $x^{\frac{1}{3}}$, $x^{\frac{1}{2}}$, $x^{\sqrt{2}}$, $\cdots$ for $x > 0$

In complex analysis, Can we define power functions?

e.g.: $i^{\frac{1}{2}}$; i^i ; π^i ; i^π

In general, z^c ; $c \in \mathbb{C}$.

Idea: In Calculus, how to understand

$x^{\sqrt{2}}$: $x > 0$

$$x^{\sqrt{2}} = (e^{\ln x})^{\sqrt{2}} = e^{\sqrt{2}\ln x}$$

$$x \to \ln x \to \sqrt{2}\ln x \to e^{\sqrt{2}\ln x}$$

Def^n: Let $c \in \mathbb{C}$

We define $z^c = e^{(c \log z)}$, $z \neq 0$

Note: $z^c = e^{c \log z}$

$= e^{c(\ln|z| + i\mathrm{Arg} z + i2n\pi)}$ $n \in \mathbb{Z}$

$= e^{c \ln|z|} e^{ci\mathrm{Arg} z} e^{ic2n\pi}$ $n \in \mathbb{Z}$

Remark:

① If c is an integer, say $c = k \in \mathbb{Z}$

$\Rightarrow$ $e^{ic2n\pi} = e^{i2nk\pi} = 1$

$\Rightarrow$ z^c is single valued

② If c is NOT an integer

$\Rightarrow$ $e^{ic2n\pi}$ may have different values for different $n \in \mathbb{Z}$

$\Rightarrow$ z^c is multi-valued.

$$z \xrightarrow{z} \boxed{z^c} \rightrightarrows$$

Def^n: (principal value of z^c)

We define the principal value of z^c:

$$\text{p.v. } z^c = e^{c \operatorname{Log} z}$$

E.g.:

(1) Find i^i and p.v. i^i.

Recall $i^i = e^{i \log i}$

For that, we compute

$$\log i = \overset{0}{\cancel{\ln|i|}} + i\,\overset{\frac{\pi}{2}}{\cancel{\operatorname{Arg}(i)}} + i2n\pi \quad n \in \mathbb{Z}$$

$$= i\frac{\pi}{2} + i2n\pi, \quad n \in \mathbb{Z}$$

$$\Rightarrow i^i = e^{i \log i} = e^{i\left(i\frac{\pi}{2} + i2n\pi\right)}$$

$$= e^{-\frac{\pi}{2} - 2n\pi}, \quad n \in \mathbb{Z}$$

Recall: P.V. $i^i = e^{i \operatorname{Log} i}$

$$\operatorname{Log} i = \ln(i) + i \operatorname{Arg}(i)$$
$$= 0 + i\frac{\pi}{2} = i\frac{\pi}{2}$$

i^i

$$\Rightarrow \text{P.V. } i^i = e^{i(i\frac{\pi}{2})}$$
$$= e^{-\pi/2}$$

(2) Find $1^{\frac{1}{\pi}}$ P.V. $1^{\frac{1}{\pi}}$

Recall : $1^{\frac{1}{\pi}} = e^{\frac{1}{\pi}\log 1}$

$= e^{\frac{1}{\pi}(\underbrace{\cancel{\ln 1}}_{0} + i\underbrace{\cancel{\mathrm{Arg}\,1}}_{0} + i2n\pi)}$

$= e^{\frac{1}{\pi}\cdot i2n\pi} = e^{2ni}$

$n \in \mathbb{Z}$

P.V. $1^{\frac{1}{\pi}} = ?$ Ex

Recall $\operatorname{Log} z$ is analytic

on $D = \{re^{i\theta} \mid r>0,\ -\pi<\theta<\pi\}$

$$\Rightarrow \quad \text{P.V. } z^c = e^{c \operatorname{Log} z} \quad \left\{ \begin{array}{l} w = \operatorname{Log} z \\ e^{cw} \end{array} \right.$$

is analytic on D

Next, how to define $\sin z$, $\cos z$, $\cosh z$, $\sinh z$

Idea: Recall for $x \in \mathbb{R}$

$$e^{ix} = \cos x + i \sin x \quad (1)$$

$$e^{-ix} = \cos(-x) + i \sin(-x)$$
$$= \cos x - i \sin x \quad (2)$$

Hence

$$(1) + (2) \Rightarrow \cos x = \frac{e^{ix} + e^{-ix}}{2} \quad (3)$$

$$(1) - (2) \Rightarrow \sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad (4)$$

Q: Can we define $\cos z$, $\sin z$ for $z \in \mathbb{C}$?

A: Yes!

Def^n: For $z \in \mathbb{C}$, define

$$\cos z \triangleq \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z \triangleq \frac{e^{iz} - e^{-iz}}{2i}$$

Remark:

① They are $\underbrace{\text{entire}}_{\text{analytic on } \mathbb{C}}$

②

$$(\cos z)' = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)'$$

$$= \frac{1}{2}\left(\left(e^{iz}\right)' + \left(e^{-iz}\right)'\right)$$

$$= \frac{1}{2}\left(e^{iz} \cdot i + e^{-iz} \cdot (-i)\right)$$

$$\boxed{\begin{array}{l}\text{In Calculus} \\ (\sin x)' = \cos x \\ (\cos x)' = -\sin x\end{array}}$$

$$\frac{d}{dz} f \circ g(z)$$

$$= f'(g(x)) \cdot g'(x)$$

$$= i \, \frac{e^{iz} - e^{-iz}}{2}$$

$$\boxed{i = \frac{-1}{i} \quad (\text{as } i^2 = -1)}$$

$$= \frac{-1}{i} \, \frac{e^{iz} - e^{-iz}}{2}$$

$$= - \frac{e^{iz} - e^{-iz}}{2i}$$

$$= - \sin z$$

E.X.:

$$(\sin z)' = \cos z$$

Recall for $x \in \mathbb{R}$,

$$\begin{cases} \cosh x = \dfrac{e^x + e^{-x}}{2} & (5) \\ \sinh x = \dfrac{e^x - e^{-x}}{2} & (6) \end{cases}$$

They satisfy:

$$\begin{cases} (\cosh x)' = \sinh x \\ (\sinh x)' = \cosh x \end{cases}$$

Q: Can we define $\cosh z$, $\sinh z$?

A: Yes!

Defn: For $z \in \mathbb{C}$, define

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

Remark:

① They are entire

② $(\cosh z)' = \sinh z$

$(\sinh z)' = \cosh z$

$$\begin{cases} \cosh(iz) = \cos z \\ \sinh(iz) = i\sin z \end{cases}$$

E.x: when $\sin z = 0$?

Recall in Calculus,

$\sin x = 0 \iff x = n\pi, \ n \in \mathbb{Z}$
$x \in \mathbb{R}$

A: $\sin z = 0 \iff$

$$e^{iz} - e^{-iz} = 0$$

$$\iff e^{iz} = e^{-iz}$$

multiple by e^{iz}

$$\iff e^{iz} \cdot e^{iz} = e^{-iz} e^{iz}$$

$$\iff e^{2iz} = e^{0} = 1$$

To be continued.