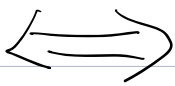


Derivative in complex analysis.

Defⁿ: Let $f: S \subseteq \mathbb{C} \rightarrow \mathbb{C}$

f is (complex) differentiable at z_0



$$(*) \quad f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \underline{\text{exists}}$$

exists as a
complex number
(NOT ∞)

$$= \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z}$$

where $\Delta w = f(z_0 + \Delta z) - f(z_0)$

In the future,

- real differentiable / \mathbb{R} -diff. ...

means "differentiable in the sense of Calculus"

- complex differentiable / \mathbb{C} -diff. ...

means "differentiable in the sense of Complex analysis"

When we simply say "differentiable",

we mean "complex differentiable"

Rule of Differentiation.

Assume f, g are differentiable. Then

Sum Rule \rightarrow ① $(f+g)'(z) = f'(z) + g'(z)$

Product Rule \leftarrow ② $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$

Quotient Rule \leftarrow ③ $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$ when $g(z) \neq 0$

④ $\frac{d}{dz}[c] = 0$ Here c : constant

power Rule \rightarrow $\frac{d}{dz}[z^n] = nz^{n-1}$ Here $n \geq 1, n \in \mathbb{Z}$

(Notation: $\frac{d}{dz} f = f'$: complex derivative)

⑤ chain rule

$$\frac{d}{dz}[f \circ g(z)] = f'(g(z)) \cdot g'(z)$$

derivative of outer function \cdot derivative of inner function

E.g.: we can differentiate polynomials and rational functions.

$$\bullet \frac{d}{dz} (z^3 - 5z^2 + i)$$

$$= \frac{d}{dz} (z^3) - \frac{d}{dz} (5z^2) + \frac{d}{dz} (i)$$

$$= 3z^2 - 10z + 0$$

$$= 3z^2 - 10z$$

$$\bullet \frac{d}{dz} \left(\frac{1}{z^2+1} \right)$$

apply quotient Rule

$$= \frac{1' (z^2+1) - 1 \cdot (z^2+1)'}{(z^2+1)^2}$$

$$= \frac{-2z}{(z^2+1)^2} \quad \text{when } z^2+1 \neq 0$$

(i.e. $z \neq \pm i$)

Recall in Calculus:

$f(x)$ real differentiable at x_0

$\Rightarrow f$ continuous at x_0

We have a similar result for complex differentiable functions

Thm: If f is differentiable at z_0 ,

$\Rightarrow f$ is continuous at z_0

(Pf: Not required)

Pf: Goal: $\lim_{z \rightarrow z_0} f(z) = f(z_0)$

By assumption

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists}$$

Note, if z close to z_0 , and $z \neq z_0$

$$f(z) - f(z_0) = \frac{f(z) - f(z_0)}{z - z_0} (z - z_0)$$

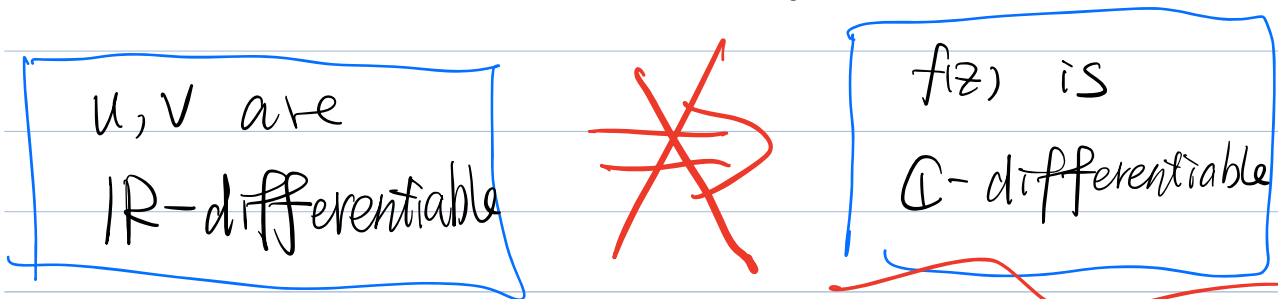
let $z \rightarrow z_0$ and take limit:

$$\lim_{z \rightarrow z_0} (f(z) - f(z_0)) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \rightarrow z_0} (z - z_0)$$

$$= f'(z_0) \cdot 0 = 0$$

Cauchy - Riemann Equations

Recall if $f(z) = u(x, y) + i v(x, y)$



(i.e. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist)

E.g. $f(z) = |z|^2 = x^2 + y^2 + i \cdot 0$

where $z = x + iy$

Then $\begin{cases} u = x^2 + y^2 \\ v = 0 \end{cases}$

In this e.g.

u, v are \mathbb{R} -diff^{ble} in \mathbb{R}^2 ,
but $f(z)$ is only \mathbb{C} -diff^{ble} at $z=0$.

... , ... , ...

But we will see :

① u, v are \mathbb{R} -differentiable

+

② $???$
 $???$



$f(z)$ is \mathbb{C} -differentiable

← Find this out today.

Recall from Calculus:

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y}$$

$$z_0 = x_0 + iy_0$$

Assume $f'(z_0)$ exists

write $f(z) = u(x, y) + i v(x, y)$

Recall $f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ exists

write $\Delta W = f(z_0 + \Delta z) - f(z_0)$

let's compute the above limit

write

$$\left\{ \begin{array}{l} \Delta W = \Delta u + i \Delta v \\ \Delta z = \Delta x + i \Delta y \end{array} \right.$$

$w = u + i v$
 $z = x + i y$

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{\Delta W}{\Delta z}$$

We pick two directions to compute limits

Way (1). If $\Delta z = \Delta x + i \cdot 0$ (i.e., $\Delta z \rightarrow 0$ along x-axis)

$$\frac{\Delta W}{\Delta z} = \frac{\Delta u}{\Delta z} + i \frac{\Delta v}{\Delta z} = \frac{\Delta u}{\Delta x} + i \frac{\Delta v}{\Delta x}$$

$$= \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

Let $\Delta z = \Delta x \rightarrow 0$ and take limit

limit of

$$\text{the above} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$= \boxed{u_x + i v_x}$$

(Notation: $u_x = \frac{\partial u}{\partial x}$, $v_x = \frac{\partial v}{\partial x}$)

limit along x-axis

Way (2):

$$\Delta z = i\Delta y$$

$$\Delta z \rightarrow 0$$

If $\Delta z = 0 + i\Delta y$ (i.e., along y-axis)

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u}{\Delta z} + i \frac{\Delta v}{\Delta z}$$

Δu

$$= \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y}$$

Δv

$$= \frac{1}{i} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} + \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y}$$

let $\Delta z \rightarrow 0$ (i.e., $\Delta y \rightarrow 0$) and take limit,

$$f'(z_0) = \frac{1}{i} u_y(x_0, y_0) + v_y(x_0, y_0)$$

(2)

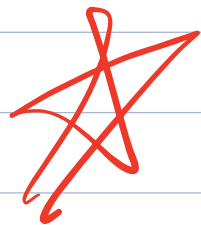
$$= v_y(x_0, y_0) - i u_y(x_0, y_0)$$

$$\frac{1}{i} = \frac{1}{i} \frac{i}{i} = -i$$

limit along y-axis

Compare (1) and (2) \Rightarrow

$$\begin{aligned} f'(z_0) &= u_x(x_0, y_0) + i v_x(x_0, y_0) \\ &= v_y(x_0, y_0) - i u_y(x_0, y_0) \end{aligned}$$



$$\Rightarrow \begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0) \\ v_x(x_0, y_0) = -u_y(x_0, y_0) \end{cases}$$

Cauchy Riemann equations

Summarize:

Thm: If $f'(z_0)$ exists,

$$\begin{array}{l} \Rightarrow \\ \text{Cauchy} \rightarrow \\ \text{Riemann eqns} \end{array} \left\{ \begin{array}{l} u_x(x_0, y_0) = v_y(x_0, y_0) \\ u_y(x_0, y_0) = -v_x(x_0, y_0) \end{array} \right.$$

and

$$\begin{aligned} f'(z_0) &= u_x(x_0, y_0) + i v_x(x_0, y_0) \\ &= v_y(x_0, y_0) - i u_y(x_0, y_0) \end{aligned}$$

Thm

Summarize:

Write $f(z) = u(x, y) + i v(x, y)$

①

u, v are \mathbb{R} -diffble
at (x_0, y_0)

+

②

$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ at (x_0, y_0)

$C, -R$ eqns

\Leftrightarrow

$f(z)$ is \mathbb{C} -diffble
at $z_0 = x_0 + iy_0$

In this case, $f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$

$$= v_y(x_0, y_0) - i u_y(x_0, y_0)$$

We proved " \Leftarrow "

" \Rightarrow " Read it by yrself

E.g $f(z) = z^2$

(A) we did last time by

defⁿ:

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= 2z$$

(B) Today: Use C.-R. eqns

Step 1: find u, v

$$f(z) = z^2 \\ = (x + iy)^2$$

$$= \underbrace{(x^2 - y^2)}_u + i \underbrace{(2xy)}_v$$

Step 2: Compute u_x, u_y
 v_x, v_y

and check C.-R. eqns
hold.

$$u_x = 2x, \quad u_y = -2y$$

$$v_x = 2y, \quad v_y = 2x$$

$$\Rightarrow \begin{cases} u_x = v_y \\ v_x = -u_x \end{cases} \text{ hold}$$

$\Rightarrow f'(z)$ exists!

$$f'(z) = u_x + i v_x$$

$$= z_x + i(z_y)$$

$$= z(x + iy)$$

$$= z z$$