Derivative in complex analysis.

Defn: Let $f: S \subseteq \mathbb{C} \to \mathbb{C}$

f is (complex) differentiable at z_0

$\Longleftrightarrow$

$$(*) \quad f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists}$$

exists as a complex number (NOT ∞)

$$= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

where $\Delta w = f(z_0 + \Delta z) - f(z_0)$

In the future,

- real differentiable / $\mathbb{R}$ - diff. . . .

  means "differentiable in the sense of Calculus"

- complex differentiable / $\mathbb{C}$-diff . . .

  means "differentiable in the sense of Complex analysis"

When we simply say "differentiable", we mean "complex differentiable"

Rule of Differentiation.

Assume f, g are differentiable. Then

Sum Rule → ① $(f+g)'(z) = f'(z) + g'(z)$

Product Rule → ② $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$

Quotient Rule → ③ $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$ when $g(z) \neq 0$

④ $\frac{d}{dz}[c] = 0$ Here c: constant

Power Rule → $\frac{d}{dz}[z^n] = nz^{n-1}$ Here $n \geq 1$, $n \in \mathbb{Z}$

(Notation: $\frac{d}{dz} f = f'$: complex derivative)

⑤ Chain rule

$$\frac{d}{dz}[f \circ g(z)] = \underbrace{f'(g(z))}_{\text{derivative of outer function}} \cdot \underbrace{g'(z)}_{\text{derivative of inner function}}$$

E.g: We can differentiate polynomials and rational functions.

- $\frac{d}{dz}(z^3-5z^2+i)$

$$= \frac{d}{dz}(z^3) - \frac{d}{dz}(5z^2) + \frac{d}{dz}(i)$$

$$= 3z^2 - 10z + 0$$

$$= 3z^2 - 10z$$

- $\frac{d}{dz}\left(\frac{1}{z^2+1}\right)$

apply quotient Rule

$$= \frac{1'(z^2+1) - 1\cdot(z^2+1)'}{(z^2+1)^2}$$

$$= \frac{-2z}{(z^2+1)^2}, \quad \text{When } z^2+1 \neq 0 \quad (\text{i.e. } z \neq \pm i)$$

Recall in Calculus:

$f(x)$ real differentiable at x_0

$\Rightarrow$ f continuous at x_0

We have a similar result for complex differentiable functions

Thm: If f is differentiable at z_0,

$\Rightarrow$ f is continuous at z_0

(Pf: NOT required)

Pf: Goal: $\lim_{z \to z_0} f(z) = f(z_0)$

By assumption

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists}$$

Note, if z close to z_0, and $z \neq z_0$

$$f(z) - f(z_0) = \frac{f(z) - f(z_0)}{z - z_0} (z - z_0)$$

Let $z \to z_0$ and take limit:

$$\lim_{z \to z_0} (f(z) - f(z_0)) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \to z_0} (z - z_0)$$

$$= f'(z_0) \cdot 0 = 0$$

Cauchy-Riemann Equations

Recall if $f(z) = u(x,y) + iv(x,y)$

$\boxed{u, v \text{ are } \mathbb{R}\text{-differentiable}} \not\Rightarrow \boxed{f(z) \text{ is } \mathbb{C}\text{-differentiable}}$

(i.e. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist)

E.g: $f(z) = |z|^2 = \underbrace{x^2+y^2}_{u} + i \cdot \underbrace{0}_{v}$

where $z = x+iy$

Then $\begin{cases} u = x^2+y^2 \\ v = 0 \end{cases}$

In this e.g,

u, v are $\mathbb{R}$-diff^{ble} in $\mathbb{R}^2$,

but $f(z)$ is only $\mathbb{C}$-diff^{ble} at $z=0$.

But we will see:

① u, v are $\mathbb{R}$-differentiable

\+

② ???

$\Longleftrightarrow$ f(z) is $\mathbb{C}$-differentiable

Find this out today.

Recall from Calculus:

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}$$

$z_0 = x_0 + iy_0$

Assume $f'(z_0)$ exists

Write $f(z) = u(x,y) + iV(x,y)$

Recall

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \text{ exists}$$

Write $\Delta w = f(z_0+\Delta z) - f(z_0)$

Let's compute the above limit

write $\begin{cases} \Delta w = \Delta u + i\Delta V \\ \Delta z = \Delta x + i\Delta y \end{cases}$

$w = u + iV$

$z = x + iy$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

We pick two directions to compute limits

Way (1): If $\Delta z = \Delta x + i \cdot 0$ (i.e., along x-axis $\Delta z \to 0$)

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u}{\Delta z} + i\frac{\Delta v}{\Delta z} = \frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x}$$

$$= \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

Let $\Delta z = \Delta x \to 0$ and take limit

limit of the above $= \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

$$= \boxed{u_x + i v_x}$$

(Notation: $u_x = \dfrac{\partial u}{\partial x}$, $v_x = \dfrac{\partial v}{\partial x}$)

limit along x-axis

Way (2):

$\Delta z = i\Delta y$

$\Delta z \to 0$

If $\Delta z = 0 + i\Delta y$ (i.e., along y-axis)

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u}{\Delta z} + i\,\frac{\Delta v}{\Delta z}$$

Δu Δv

$$= \frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{i\Delta y} + i\,\frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{i\Delta y}$$

$$= \frac{1}{i}\,\frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{\Delta y} + \frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{\Delta y}$$

Let $\Delta z \to 0$ (i.e., $\Delta y \to 0$) and take limit,

$-i$

$$f'(z_0) = \frac{1}{i}\,u_y(x_0, y_0) + v_y(x_0, y_0)$$

(2)
$$= v_y(x_0, y_0) - i\,u_y(x_0, y_0)$$

$$\boxed{\frac{1}{i} = \frac{1}{i}\,\frac{i}{i} = -i}$$

Limit along y-axis

Compare (1) and (2) $\Longrightarrow$

$$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$$
$$= v_y(x_0, y_0) - i u_y(x_0, y_0)$$

$$\Rightarrow \begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0) \\ v_x(x_0, y_0) = -u_y(x_0, y_0) \end{cases}$$

Cauchy Riemann equations

Summarize:

Thm: If $f'(z_0)$ exists,

$$\Rightarrow \begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0) \\ u_y(x_0, y_0) = -v_x(x_0, y_0) \end{cases}$$

Cauchy Riemann eqns $\rightarrow$

and

$$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$$

$$= v_y(x_0, y_0) - i u_y(x_0, y_0)$$

Thm

Summarize :

Write $f(z) = u(x,y) + iv(x,y)$

① $\boxed{u, v \text{ are } \mathbb{R}\text{-diff}^{ble} \text{ at } (x_0, y_0)}$

+

② $\boxed{\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \text{ at } (x_0, y_0)}$

C.-R. eqns

$\Longleftrightarrow$ $\boxed{f(z) \text{ is } \mathbb{C}\text{-diff}^{ble} \text{ at } z_0 = x_0 + iy_0}$

In this case, $f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$

$= v_y(x_0, y_0) - i u_y(x_0, y_0)$

We proved "$\Longleftarrow$"

" $\Longrightarrow$ " Read it by urself

E.g $\quad f(z) = z^2$

(A) we did last time by defn:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= 2z$$

(B) Today: Use C.-R. eqns

Step 1: find u, v

$$f(z) = z^2$$
$$= (x+iy)^2$$
$$= \underbrace{(x^2-y^2)}_{u} + i\underbrace{(2xy)}_{v}$$

Step2: Compute u_x, u_y
v_x, v_y
and check C.-R. eqns hold.

$u_x = 2x, \quad u_y = -2y$

$v_x = 2y, \quad v_y = 2x$

$$\Rightarrow \begin{cases} u_x = v_y \\ v_x = -u_x \end{cases} \quad \text{hold}$$

$\Rightarrow$ $f'(z)$ exists !

$$f'(z) = u_x + i v_x$$
$$= 2x + i(2y)$$
$$= 2(x + iy)$$
$$= 2z$$