Problem 1:

In each case, write the function \( f(z) \) in the form \( f(z) = u(x, y) + i v(x, y) \):

(b) \( f(z) = \frac{z^2}{z} \) \( (z \neq 0) \). \( \text{Suggestion: In part (b), start by multiplying the numerator and denominator by } \bar{z}. \)

\[
f(z) = \frac{z^2}{z} = \frac{z^2}{z} = \frac{z^3}{2i^2} \quad \text{Let } z = x + iy, \quad \bar{z} = x - iy
\]

\[
f(z) = \frac{(x - iy)^3}{x^2 + y^2} = \frac{x^3 - 3i(x^2y - y^2) + 3i^2xy - i^3y^3}{x^2 + y^2} = \frac{x^3 + iy^3 - 3x^2y + 3i(x^2y - y^2)}{x^2 + y^2}
\]

where \( u(x, y) = \frac{x^3 - 3x^2y}{x^2 + y^2}, u(x, y) = \frac{y^3 - 3x^2y}{x^2 + y^2} \) \( \text{Answer} \).

Problem 2:

4. Write the function

\[ f(z) = z + \frac{1}{z} \quad (z \neq 0) \]

in the form \( f(z) = u(r, \theta) + i v(r, \theta) \).

\[ \text{Let } z = re^{i\theta} = r(\cos \theta + i \sin \theta) \text{ then} \]

\[
f(z) = z + \frac{1}{z} = re^{i\theta} + \frac{1}{re^{i\theta}} = re^{i\theta} + \frac{r}{re^{i\theta}} = re^{i\theta} + \frac{r}{2} e^{-i\theta}
\]

\[
= r \cos \theta + \frac{r \cos \theta}{2} + i \sin \theta \left( \frac{r - \frac{r}{2}}{r} \right)
\]

So comparing real & imaginary parts:

\( u(r, \theta) = (r + \frac{r}{2}) \cos \theta \)

\( v(r, \theta) = \sin \theta (r - \frac{r}{2}) \)
Problem 3:

5. By referring to the discussion in Sec. 14 related to Fig. 19 there, find a domain in the $z$ plane whose image under the transformation $w = z^2$ is the square domain in the $w$ plane bounded by the lines $u = 1, u = 2, v = 1,$ and $v = 2.$ (See Fig. 2, Appendix 2.)

5) Let $f(z) = z^2.$ Let’s try to find the pre-image $S$ of the line $u = 1$ under $f$. Now, $z = x + iy \in S,$ iff $Re(f(z)) = 1.$ That is equivalent to $1 = Re(z^2) = Re(x^2 - y^2 + 2xyi) = x^2 - y^2.$ So, it corresponds to the hyperbola given by $x^2 - y^2 = 1.$ Similarly we can find the pre-images of lines $u = 2,$ $v = 1$ and $v = 2$ to be given by $x^2 - y^2 = 2,$ $xy = \frac{1}{2}$ and $xy = 1$ respectively.

Now, it is easy to see that the pre-image of the region bounded by those lines (say $R$) are two open connected sets (hence domains), bounded by above 4 hyperbolas on first and third quadrants respectively. Let’s call them $S_1$ and $S_2.$ So, $f(S_1 \cup S_2) = R.$ Observe that they are symmetric with respective to the origin. That is $S_1 = -S_2.$ Also, $f(z) = z^2 = (-z)^2 = f(-z)$ for each $z.$ Hence, in fact $f(S_1) = f(S_2).$ Therefore, $f(S_1) = f(S_1) \cup f(S_2) = f(S_1 \cup S_2) = R.$ So, $S_1$ satisfies the desired conditions.

Problem 4:

6. Find and sketch, showing corresponding orientations, the images of the hyperbolas

$$x^2 - y^2 = c_1 \ (c_1 < 0) \text{ and } 2xy = c_2 \ (c_2 < 0)$$

under the transformation $w = z^2$.

We note, here $c_1, c_2 < 0 \quad \therefore W(z) = z^2 = x^2 - y^2 + 2i xy = u(\Re(z) + i \Im(z))$.

So $x^2 - y^2 = c_1 \Rightarrow y^2 - x^2 = -c_1 \quad \text{(we did this because } -c_1 > 0\text{)}$

Similarly, $2xy = c_2 \Rightarrow -2xy = -c_2$

\[ W = z^2 \]

\[ z = x + iy \]

\[ x = \Re(z) \quad y = \Im(z) \]

\[ W(z) = z^2 = x^2 - y^2 + 2i xy = u(\Re(z) + i \Im(z)) \]
we note: if \( U(x,y) = x^2 - y^2 = c_1 \) then image of the upper branch of hyperbola of Fig 1 is expressed parametrically as:

\[
U = c_1, \quad V = 2x \sqrt{x^2 - c_1} \quad (-\infty < x < \infty)
\]

It is evident that image of \((x,y)\) moves upwards along \(U = c_1\) as we move to the right in the upper branch of hyperbola \(y^2 - x^2 = c_1\).

Similarly, for lower half of hyperbola, a parametrisation is:

\[
U = c_1, \quad V = -2x \sqrt{x^2 - c_1} \quad (-\infty < x < \infty)
\]

We see image of a point \((x,y)\) goes upwards along \(U = c_1\) as we go in left direction in this lower branch of hyperbola. (-x direction)

Now, if \( U(x,y) = 2xy = c_2 \quad (<0) \) then image of the branch in second quadrant can be written parametrically as:

\[
U(x,y) = x^2 - \left(\frac{c_2}{2x}\right)^2 = x^2 - \frac{c_2^2}{4x^2} \quad , \quad U(x,y) = c_2 \quad (-\infty < x < 0)
\]

Because here \( y = \frac{c_2}{2x} \) & \( U(x,y) = 2xy \).

Now we note...

let \( U(x,y) = c_2 \quad (<0) \)
Now we note:
\[ u(x, y) = +\infty \quad \text{as} \quad x \to 0 \quad y \to 0 \]
\[ u(x, y) = -\infty \quad \text{as} \quad x \to 0 \quad y \to 0 \]

and since \( u(x, y) \) varies continuously with \( x \) \& \( y \) hence as we go towards \( x = 0 \) line along the branch of \( 2y = c_2 \) in the second quadrant, the image of the point \((x, y)\) goes towards the left direction along \( 2y = c_2 \).

Similarly, image of branch \( u(x, y) = c_2 \) in the 4th quadrant can be written parametrically as (same as above)
\[ u(x, y) = x^2 = \frac{c_2}{4} x^2 \quad , \quad u(x, y) = c_2 (< 0) \quad 0 < x < \infty \]

we note \[ \lim_{x \to 0^-} u(x, y) = +\infty \quad \text{&} \quad \lim_{x \to 0^+} u(x, y) = -\infty \]

So here also as we go towards \( x = 0 \) line in Fig 1, along the branch of \( 2y = c_2 \) in 4th quadrant, image of \((x, y)\) goes towards left direction along \( 2y = c_2 \) line.