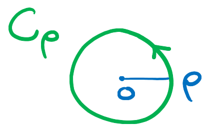
14. The Cauchy Integral Formula

① "Keyhole Surgery"

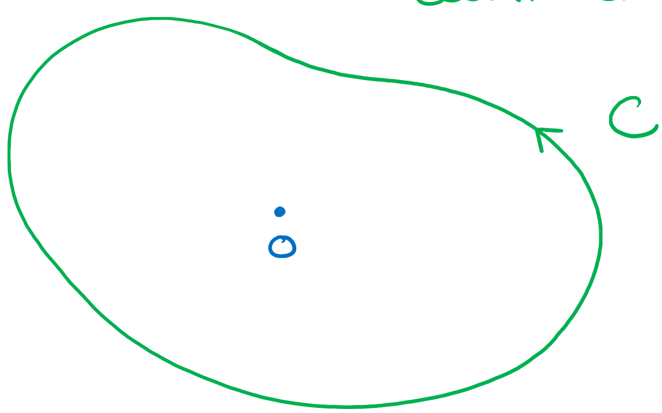


Recall: Given $C_\rho : z(t) = \rho e^{it}$, $0 \le t \le 2\pi$

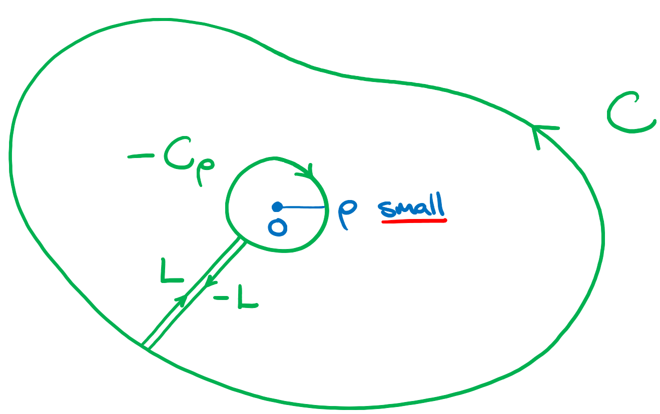
$$\int \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{\cancel{\rho e^{it}}} \cdot i\cancel{\rho e^{it}}\, dt = 2\pi i,$$

independent of the radius ρ of the circle.

Now, let C be a simple closed contour around 0 in the positive direction.
counterclockwise



Problem: Compute $\int_C \frac{1}{z}\, dz$.



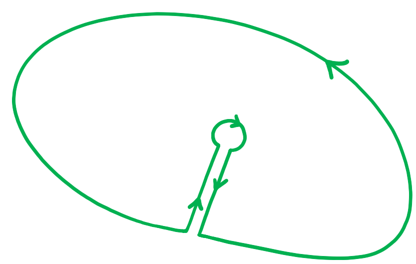
Apply Cauchy-Goursat thm. to "$C + L - C_\rho - L$" to get

$$\int_C \frac{1}{z}\,dz + \int_L \frac{1}{z}\,dz + \underbrace{\int_{-C_\rho} \frac{1}{z}\,dz}_{= -\int_{C_\rho} \frac{1}{z}\,dz} + \underbrace{\int_{-L} \frac{1}{z}\,dz}_{= -\int_L \frac{1}{z}\,dz} = 0.$$

$$\rightsquigarrow \quad \int_C \frac{1}{z}\,dz = \int_{C_\rho} \frac{1}{z}\,dz = 2\pi i.$$

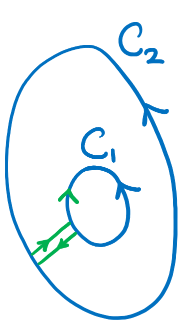
Informal terminology:

"keyhole contour"



For analytic functions more generally we have:

Theorem ("Principle of deformations of paths"):



Let C_1 and C_2 denote positively oriented simple closed contours, with C_1 inside C_2. If $f(z)$ is analytic on C_1 and C_2 and in between, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Multiple keyhole surgeries:

Theorem:

C simple closed, pos. oriented.

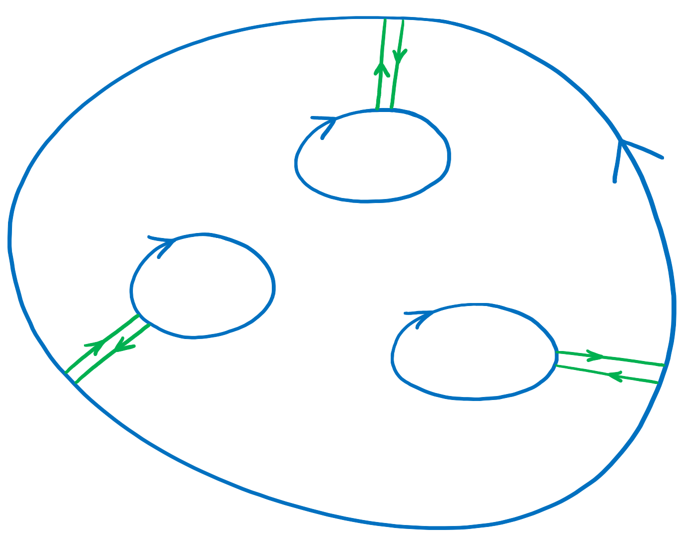
$C_1, \ldots, C_n$ simple closed, inside C, neg. oriented.

$f(z)$ analytic on $C, C_1, \ldots, C_n$ and in the region consisting of points inside C and outside $C_1, \ldots, C_n$,

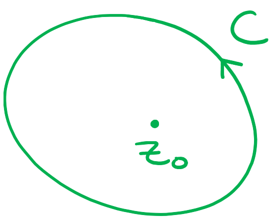
then

$$\underbrace{\int_C f(z)\,dz}_{\text{+ve orientation}} + \underbrace{\int_{C_1} f(z)\,dz}_{\text{-ve orientation}} + \cdots + \underbrace{\int_{C_n} f(z)\,dz}_{\text{-ve orientation}} = 0.$$

← Just apply Cauchy-Goursat, as before.



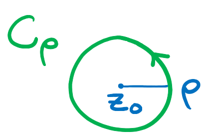
② Cauchy Integral Formula



Theorem (Cauchy Integral Formula):
f analytic inside and on a simple closed contour C which is oriented positively, and z_0 interior to C, then

$$\boxed{f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz.}$$

To see this we need an easy lemma:



Lemma: $\displaystyle \int_{C_\rho} \frac{1}{z - z_0}\, dz = 2\pi i,$

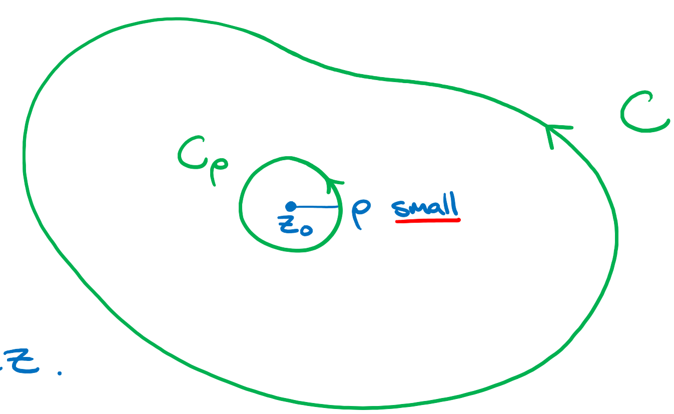
$C_\rho : \quad z(t) = z_0 + \rho e^{it}, \quad 0 \le t \le 2\pi.$

$$\int_{C_\rho} \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{1}{\rho e^{it}}\, i\rho e^{it}\, dt = \int_0^{2\pi} i\, dt = 2\pi i.$$

Proof of theorem:

By the "principle of deformations of paths"

$$\int_C \frac{f(z)}{z - z_0}\, dz = \int_{C_\rho} \frac{f(z)}{z - z_0}\, dz.$$



Now $\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_\rho} \frac{f(z)}{z-z_0}\,dz$ holds for any $\rho > 0$. For $\rho > 0$ very small, $f(z)$ will be very close to $f(z_0)$, so $\int_{C_\rho} \frac{f(z)}{z-z_0}\,dz$ will be very close to

$$\int_{C_\rho} \frac{f(z_0)}{z-z_0}\,dz = f(z_0)\int_{C_\rho} \frac{dz}{z-z_0} = 2\pi i\, f(z_0).$$

$$\rightsquigarrow \quad \int_{C_\rho} \frac{f(z)}{z-z_0}\,dz \to 2\pi i\, f(z_0) \quad \text{as } \rho \to 0. \qquad (*)$$

this should really be done more carefully, see below.

But since $\int_{C_\rho} \frac{f(z)}{z-z_0}\,dz$ does not depend on ρ, we must have

$$\int_{C_\rho} \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0).$$

$\square$

Formal proof of $(*)$:

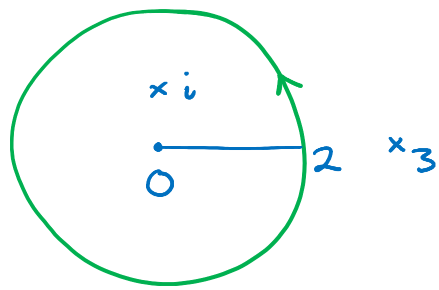
$$\int_{C_\rho} \frac{f(z)}{z-z_0}\,dz - \int_{C_\rho} \frac{f(z_0)}{z-z_0}\,dz = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz.$$

$$\frac{f(z)-f(z_0)}{z-z_0} = f'(z_0) + \psi(z), \quad \lim_{z\to z_0} \psi(z) = 0.$$

$$\left|\int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \le \Big(\underbrace{|f'(z_0)|}_{\text{const.}} + \underbrace{\max_{|z|=\rho} |\psi(z)|}_{\to 0 \text{ as } \rho\to 0}\Big)\underbrace{2\pi\rho}_{\to 0 \text{ as } \rho\to 0} \to 0 \text{ as } \rho \to 0$$

Example: C: circle of radius 2 in positive (counterclockwise) direction.
Evaluate

$$\int_C \frac{z^2}{(z-3)^2(z-i)}\,dz.$$



Take $f(z) = \frac{z^2}{(z-3)^2}$, which is analytic on $\mathbb{C} - \{3\}$. Then $f(z)$ is analytic on C and inside C. Hence we may apply the Cauchy integral formula, with $z_0 = i$.

$$\rightsquigarrow \int_C \frac{z^2}{(z-3)^2(z-i)}\,dz = \int_C \frac{f(z)}{z-i}\,dz$$

$$= 2\pi i\, f(i).$$

$$f(i) = \frac{i^2}{(i-3)^2} = \frac{-1}{i^2-6i+9} = \frac{1}{6i-8}$$

$$\rightsquigarrow \int_C \frac{z^2}{(z-3)^2(z-i)}\,dz = \frac{\pi i}{3i-4}.$$