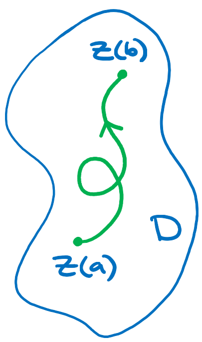
12. Antiderivatives and Path Independence

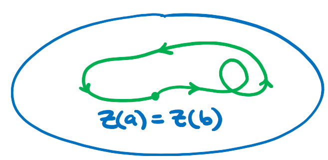


Lemma: Let $f(z)$ be a continuous function on a domain $D \subseteq \mathbb{C}$, and suppose $f(z)$ has an antiderivative $F(z)$ on D (meaning $F(z)$ is an analytic function s.t. $F'(z) = f(z)$ for all $z \in D$). Let $C: z(t),\ a \le t \le b$ be a contour. Then

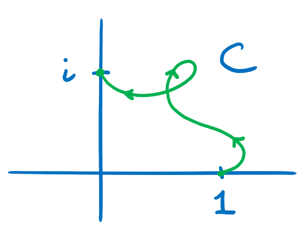
$$\boxed{\int_C f(z)\,dz = F(z(b)) - F(z(a)).}$$

Depends only on $z(a)$ & $z(b)$.

In particular, if C is closed $(z(a) = z(b))$ then $\int_C f(z)\,dz = 0$.



Examples:

① 

$$\int_C z\,dz = \left[\frac{z^2}{2}\right]_1^i = \frac{i^2}{2} - \frac{1}{2} = -1.$$

② Same contour.

$$\int_C (z^2+i)\,dz = \left[\frac{z^3}{3} + iz\right]_1^i$$
$$= \left(\frac{-i}{3} - 1\right) - \left(\frac{1}{3} + i\right)$$
$$= -\frac{4}{3} - \frac{4}{3}i.$$

Proof of lemma:

Chain Rule: $\frac{d}{dt}F(z(t)) = F'(z(t))\,z'(t)$.

Why?

$$\frac{F(z(t+\Delta t)) - F(z(t))}{\Delta t} = \frac{F(z(t+\Delta t)) - F(z(t))}{z(t+\Delta t) - z(t)} \cdot \frac{z(t+\Delta t) - z(t)}{\Delta t}$$

$\downarrow \Delta t \to 0$ $\qquad\qquad \downarrow \qquad\qquad \downarrow$

$\frac{d}{dt}F(z(t)) \qquad F'(z(t)) \qquad z'(t)$

Hence

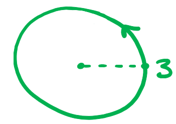
$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt = \int_a^b F'(z(t))\,z'(t)\,dt$$
$$= \int_a^b \frac{d}{dt}\left[F(z(t))\right]dt \underset{\text{FTC}}{=} F(z(b)) - F(z(a)).$$

$\square$

Examples:

① $C: \quad z(t) = 3e^{it}, \quad 0 \le t \le 2\pi.$

Compute $\int_C \frac{1}{z^2}\, dz.$



Since $\frac{1}{z^2} = \left(-\frac{1}{z}\right)'$ and C is closed,

$$\int_C \frac{1}{z^2}\, dz = 0.$$

Similarly $\int_C \frac{1}{z^n}\, dz = 0$ for all integers $n \ge 2$.

$\rightsquigarrow$ What about $n = 1$?

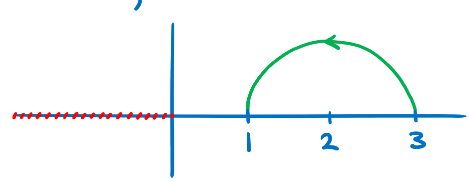
② Same contour. What about $\int_C \frac{1}{z}\, dz$?

$$\int_C \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{\cancel{3e^{it}}} \cdot \cancel{3}i\cancel{e^{it}}\, dt = 2\pi i.$$

$$\boxed{\frac{1}{z} \text{ does } \underline{\text{not}} \text{ admit an antiderivative in } \mathbb{C} - \{0\}.}$$

③ $C: \quad z(t) = 2 + e^{it}, \quad 0 \le t \le \pi.$

Compute $\int_C \frac{1}{z}\, dz.$



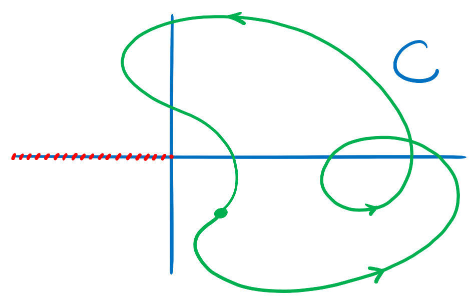
Let $D = \{ z \in \mathbb{C} - \{0\} : -\pi < \operatorname{Arg} z < \pi \}.$

$\mathrm{Log}\, z$ is analytic in D and $(\mathrm{Log}\, z)' = \frac{1}{z}$,

so $\displaystyle\int_C \frac{1}{z}\, dz = \Big[\mathrm{Log}\, z\Big]_3^1 = \ln 1 - \ln 3$

$= -\ln 3.$

(4)



$$\int_C \frac{1}{z}\, dz = 0.$$

C is closed and lies in D.

What about the converse of the lemma?

If the contour integrals of $f(z)$ do not depend on the path (in D) then does $f(z)$ have an antiderivative?

Note:

path independence of contour integrals $\Longleftrightarrow$ integrals around all closed contours vanish

Brown & Churchill p. 141 $\longrightarrow$ Theorem: $f(z)$ continuous in a domain D.

(i) $f(z)$ admits an antiderivative $F(z)$ in D.

$\Updownarrow$

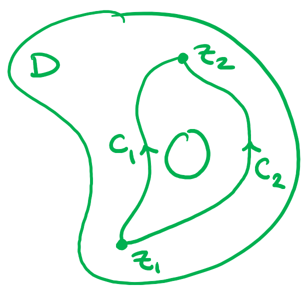
(ii) $\int_C f(z)\, dz = 0$ for all closed contours C lying entirely in D.

That (i) $\Rightarrow$ (ii) follows from the lemma.

We skip (ii) $\Rightarrow$ (i).

Basic idea:

Since $\int_C f(z)\,dz = 0$ for any closed contour C in D, $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ for any two contours C_1 & C_2 starting at a given point $z_1 \in D$ and ending at $z_2 \in D$.



$C = C_1 - C_2$ is closed,

so $\int_{C_1 - C_2} f(z)\,dz = 0$, hence

$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$.

We fix some point $z_0 \in D$ and define $F(z)$ by

$$F(z) = \underbrace{\int_{z_0}^{z} f(s)\,ds}_{\text{a contour integral taken over any path from } z_0 \text{ to } z \text{ (in D)}}.$$

"dummy variable" for the contour integral

We then just check that $F'(z) = f(z)$, this is very similar to the proof of FTC2 (the "Second Fund. Thm. of Calc.") from ordinary real variable calculus.