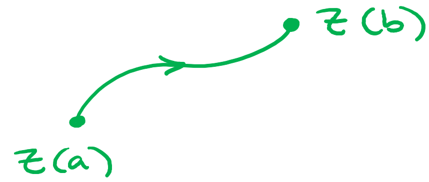
11. Contour Integrals

Contour $C: \; z(t), \quad a \le t \le b$



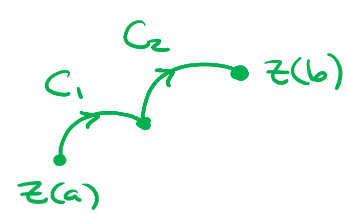
Function $f(z)$ defined on C,
and (piecewise) continuous on C.

Contour Integral:

$$\boxed{\int_C f(z)\,dz \overset{\text{def}^n}{=} \int_a^b f(z(t))\, z'(t)\,dt}$$

Remarks:

① If $C = C_1 + C_2$



then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

We compute contour integrals over piecewise smooth contours by computing the parts separately.

Note that we are implicitly using

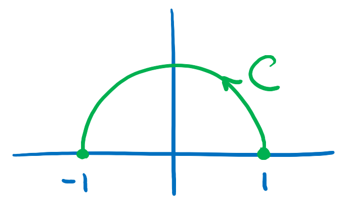
$$dz = \frac{dz}{dt}\,dt = z'(t)\,dt$$

to get $\displaystyle\int_C f(z)\,dz = \int_a^b f(z(t))\,\underbrace{z'(t)\,dt}_{dz}.$

Examples:

(1) $C: \; z(t) = e^{it}, \quad 0 \le t \le \pi.$

Compute $\displaystyle\int_C z\,dz.$



$$\int_C z\,dz = \int_0^\pi f(z(t))\,z'(t)\,dt = \int_0^\pi e^{it}\cdot ie^{it}\cdot dt$$

$$= \int_0^\pi ie^{2it}\,dt = \left[\tfrac{1}{2}e^{2it}\right]_0^\pi$$

$$= \tfrac{1}{2}\left(e^{2i\pi} - e^{2i0}\right) = \tfrac{1}{2}(1-1) = 0.$$

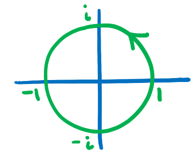
(2) Same contour. Compute $\displaystyle\int_C \bar{z}\,dz.$

$$\int_C \bar{z}\,dz = \int_0^\pi e^{-it}\cdot ie^{it}\cdot dt = \int_0^\pi i\,dt$$

$$= \left[it\right]_0^\pi = i\pi.$$

③ $C: \quad z(t) = e^{it}, \quad 0 \le t \le 2\pi.$

Compute $\int_C \frac{1}{z}\, dz.$

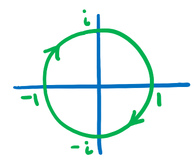


$$\int_C \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{e^{it}} \cdot i e^{it} \cdot dt = \int_0^{2\pi} i \, dt$$

$$= \left[it \right]_0^{2\pi} = \underline{\underline{2\pi i}}.$$

④ $\mathcal{C}: \quad z(t) = e^{-it}, \quad 0 \le t \le 2\pi$

Compute $\int_{\mathcal{C}} \frac{1}{z}\, dz.$



$$\int_{\mathcal{C}} \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{e^{-it}} \cdot (-ie^{-it}) \cdot dt$$

$$= \int_0^{2\pi} (-i)\, dt = -2\pi i$$

Note that $\mathcal{C} = -C$ (reversed orientation).

$\rightsquigarrow$ In general, for any contour C,

$$\boxed{\int_{-C} f(z)\, dz = -\int_C f(z)\, dz.}$$

Proof: subst. $\tau = -t$.

⑤ $C: \quad z = e^{it}, \quad 0 \le t \le 4\pi$ (going round the unit circle twice, counterclockwise)

$$\int_C \frac{1}{z}\, dz = \int_0^{4\pi} \frac{1}{e^{it}}\, i e^{it} dt = \int_0^{4\pi} i\, dt = 4\pi i.$$

Comment: Complex contour integrals and real line integrals:

$$C: \quad z(t) = x(t) + iy(t), \quad a \le t \le b$$

$$f(z) = u(x,y) + iv(x,y)$$

$$dz = dx + i\,dy$$

$$\rightsquigarrow \int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

$$= \underbrace{\int_C u\,dx - v\,dy} + i \underbrace{\int_C v\,dx + u\,dy}$$

ordinary (real) line integrals!

Recall:

$$\int_C u\,dx - v\,dy = \int_a^b \underbrace{(ux' - vy')}_{u(x(t),y(t))\,x'(t) - v(x(t),y(t))\,y'(t)}\,dt;$$

$$\int_C v\,dx + u\,dy = \int_a^b (vx' + uy')\,dt.$$

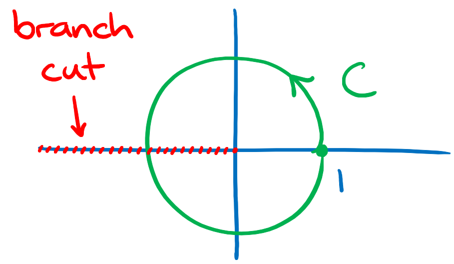
We will make use of this point of view later.

Examples involving branch cuts

Suppose we want to integrate

$f(z) = \text{P.V. } z^{1/2} = \exp\left(\frac{1}{2} \operatorname{Log} z\right)$

over the contour $C: z(t) = e^{it}, \ 0 \le t \le 2\pi$.



Does the branch cut pose a problem?

No problem!

Use this approach in general.

Option 1: Break integral into 2 parts:

$0 \le t \le \pi$ and $\pi < t \le 2\pi$.

Option 2: Since contour integrals do not depend on the parametrization (only the orientation) we could just write C as $z(t) = e^{it}$, $-\pi \le t \le \pi$.

$$f(z(t)) = \exp\left(\tfrac{1}{2} \operatorname{Log} e^{it}\right) = \exp\left(\tfrac{1}{2} it\right)$$

for $-\pi \underline{\underline{<}}\, t \le \pi$.

We do not care about the value at $t = -\pi$ as it will not affect the integral.

$$\int_C f(z)\,dz = \int_{-\pi}^{\pi} e^{\frac{it}{2}} \cdot ie^{it}\,dt = \int_{-\pi}^{\pi} ie^{\frac{3it}{2}}\,dt$$

$$= \left[\frac{2}{3} e^{\frac{3it}{2}} \right]_{-\pi}^{\pi} = \frac{2}{3}\left(e^{i\frac{3\pi}{2}} - e^{-i\frac{3\pi}{2}} \right)$$

$$= \frac{2}{3}(-i-i) = -\frac{4}{3}i.$$

Upper bounds for moduli of contour integrals

Another very useful inequality:

C contour, if $|f(z)| \le M \quad \forall z \in C$

and $\text{Length}(C) = L$ then

$$\boxed{\left| \int_C f(z)\,dz \right| \le ML.}$$

Proof:

$$\left| \int_C f(z)\,dz \right| = \left| \int_a^b f(z(t))\, z'(t)\,dt \right|$$

$$\le \int_a^b |f(z(t))|\,|z'(t)|\,dt$$

$$\le \int_a^b M\,|z'(t)|\,dt = M\int_a^b |z'(t)|\,dt$$

$$= ML. \qquad \square$$

Examples:

① $C: \ z(t) = 3e^{it}, \quad 0 \le t \le \frac{\pi}{2}.$

$\rightsquigarrow \ \text{Length}(C) = 3 \times \frac{\pi}{2} = \frac{3\pi}{2}.$

Find a bound for $\left|\int_C (\bar{z}^2+i)\,dz\right|$.

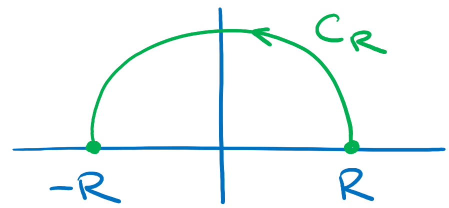
For $z \in C$, $|\bar{z}^2| = |\bar{z}|^2 = |z|^2 = 3^2 = 9$,

$\rightsquigarrow$ $|\bar{z}^2 + i| \le |\bar{z}^2| + |i| = 9 + 1 = 10$.

$\rightsquigarrow$ $\left|\int_C (\bar{z}^2+i)\,dz\right| \le \frac{30\pi}{2} = 15\pi$.

② C_R: $z(t) = Re^{it}$, $0 \le t \le \pi$,

where $R > 1$.



Claim: $$\lim_{R\to\infty} \int_{C_R} \frac{z-2}{z^4+1}\,dz = 0.$$

Proof: Let $I_R = \int_{C_R} \frac{z-2}{z^4+1}\,dz$.

Note that $\text{Length}(C_R) = \pi R$.

Recall: $|z_1 + z_2| \le |z_1| + |z_2|$,

$|z_1 + z_2| \ge \big||z_1| - |z_2|\big|$.

For $z \in C$, $|z-2| \le |z| + 2 = R+2$,

$|z^4+1| \ge \big||z^4| - 1\big| = R^4 - 1$.

$\rightsquigarrow$ $\left|\frac{z-2}{z^4+1}\right| \le \frac{R+2}{R^4-1}$ for $z \in C$.

$\rightsquigarrow$ $|I_R| \le \frac{R+2}{R^4-1} \times \pi R = \pi \frac{R^2+2R}{R^4-1}$.

We get

$$|I_R| \le \pi \frac{R^2+2R}{R^4-1} = \pi \frac{1/R^2 + 2/R^3}{1 - 1/R^4} \longrightarrow 0 \text{ as } R \to \infty.$$

multiply numerator and denominator by $\frac{1}{R^4}$

Hence $\lim\limits_{R\to\infty} I_R = 0.$ $\square$