9. Elementary Functions

① The complex exponential function

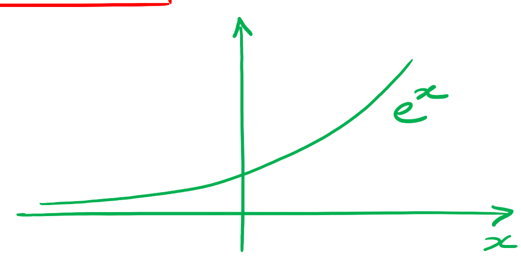
$$e^z = e^x e^{iy} = e^x \cos y + i e^x \sin y,$$

where $z = x + iy$.

e^z is an entire function

$$\boxed{\frac{d}{dz}(e^z) = e^z}$$

$$|e^z| = e^x > 0$$



$$\boxed{e^{z_1} e^{z_2} = e^{z_1+z_2}}$$

hence
$e^z e^{-z} = 1$

$$\boxed{\frac{1}{e^z} = e^{-z}}$$

$$\boxed{\arg(e^z) = y + 2n\pi,} \quad n \in \mathbb{Z}.$$

Example: Find all z s.t. $e^z = -2$.

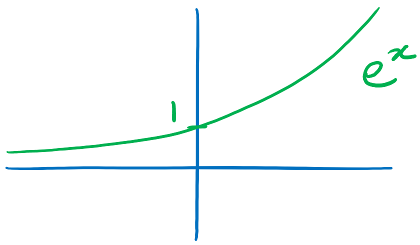
$\rightsquigarrow$ solve $e^x e^{iy} = -2 = 2e^{i\pi}$

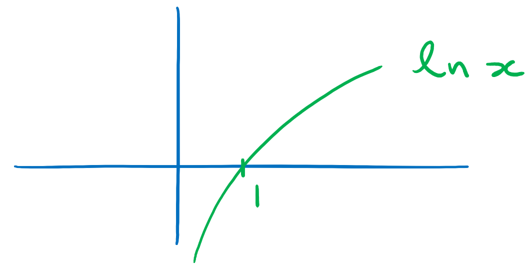
$\rightsquigarrow$ $e^x = 2$, $y = \pi + 2n\pi$, $n \in \mathbb{Z}$.

solution: $z = \ln 2 + (2n+1)\pi i, \ n \in \mathbb{Z}.$

② Complex logarithms

For $x \in \mathbb{R}$, $y = e^x \iff x = \ln y$.





Notation: we will continue to use "ln" to denote the real-variable natural logarithm.

$\rightsquigarrow$ What is the complex analog?

Given $z \in \mathbb{C}$, $z \neq 0$, find all $w \in \mathbb{C}$ s.t. $e^w = z$.

Write $z = re^{i\theta}$, $\theta = \operatorname{Arg} z$ $(-\pi < \theta \leq \pi)$

& $w = u + iv$.

$\rightsquigarrow$ $e^w = e^u e^{iv} = z = re^{i\theta}$

$\rightsquigarrow$ $\underbrace{u = \ln r}_{u = \ln|z|}$, $\underbrace{v = \theta + 2n\pi}_{v = \underline{\underline{\arg z}}}$, $n \in \mathbb{Z}$.

multivalued!

$$\boxed{e^w = z \iff w = \ln|z| + i(\arg z)}$$

Def^n: The <u>complex logarithim</u> is the <u>multivalued</u> function defined by

$$\boxed{\log z = \ln|z| + i(\arg z).}$$

Recall that while $\ln|z|$ is single valued, $\arg z$ is multivalued and is given by

$$\arg z = \underline{\operatorname{Arg} z} + 2n\pi, \quad n \in \mathbb{Z}.$$

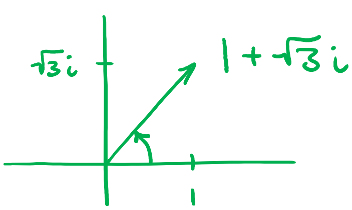
↑ Principal Argument

$-\pi < \operatorname{Arg} z \leq \pi$

Def^n: The <u>Principal Value</u> of the complex logarithm is the function

$$\boxed{\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z.}$$

Example: Find $\log(1+\sqrt{3}i)$ & $\operatorname{Log}(1+\sqrt{3}i)$.



$$1+\sqrt{3}i = 2e^{i\frac{\pi}{3}}$$

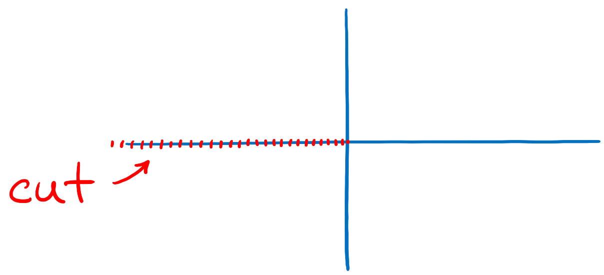
$$\rightsquigarrow \operatorname{Log}(1+\sqrt{3}i) = \ln 2 + i\frac{\pi}{3}$$

$$\& \quad \log(1+\sqrt{3}i) = \ln 2 + i\left(2n+\tfrac{1}{3}\right)\pi, \quad n \in \mathbb{Z}.$$

Question: Is $\operatorname{Log} z$ analytic?

We now consider Log z on the domain

$D = \{ re^{i\theta} : r > 0, -\pi < \theta < \pi \}$.



$z = re^{i\theta}$,

$-\pi < \theta < \pi$

$\underline{\operatorname{Log} z = \ln r + i\theta}$

Writing $f(z) = \operatorname{Log} z = u(r,\theta) + iv(r,\theta)$,

$u(r,\theta) = \ln r$, $\quad v(r,\theta) = \theta$.

Recall the C.R. eqns: $\quad ru_r = v_\theta$, $rv_r = -u_\theta$.

We have

$$ru_r = r(\ln r)' = 1 \quad \& \quad v_\theta = 1 \quad \checkmark$$

$$rv_r = 0 \quad \& \quad u_\theta = 0 \quad \checkmark$$

$\rightsquigarrow$ Log z is analytic on D.

Recall that $f'(z) = e^{-i\theta}(u_r + iv_r)$, so

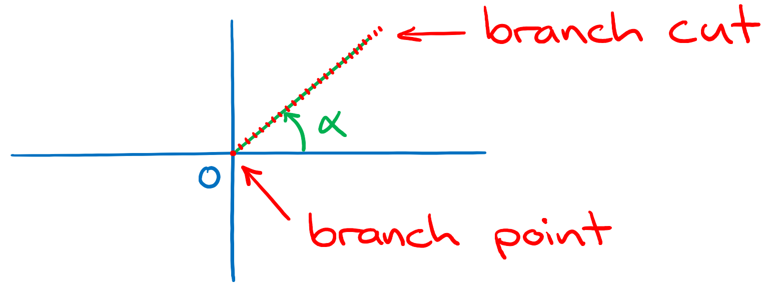
$$(\operatorname{Log} z)' = e^{-i\theta}\left(\frac{1}{r}\right) = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

$$\boxed{(\operatorname{Log} z)' = \frac{1}{z}}$$

Other branches of the logarithm:

Let $\alpha \in \mathbb{R}$. The corresponding branch of $\log z$ is the function defined on $D_\alpha = \{re^{i\theta} : r>0,\ \alpha < \theta < \alpha+2\pi\}$ by

$$(*) \qquad F(z) = \ln r + i\theta, \quad \alpha < \theta < \alpha + 2\pi.$$



The same calculation as above shows that $F'(z) = \frac{1}{z}$.

When it is clear from the context that we are working with a fixed branch of the logarithm we will simply use the notation $\log z$ for $F(z)$ as in $(*)$.

For any value of $\log z$, $z \neq 0$, we have

$$\boxed{e^{\log z} = z.}$$

What about $\log(e^z)$?

$$e^z = e^x e^{iy}, \qquad \ln|e^z| = \ln(e^x) = x,$$
$$\arg e^z = y + 2n\pi, \quad n \in \mathbb{Z}.$$

$$\rightsquigarrow \quad \log(e^z) = x + i(y + 2n\pi)$$
$$= x + iy + 2in\pi, \quad n \in \mathbb{Z}.$$

$$\boxed{\log(e^z) = z + 2in\pi,} \quad n \in \mathbb{Z}.$$

How about $\operatorname{Log}(e^z)$?

<u>Example:</u> Take $z = 2i\pi$.

$$\rightsquigarrow \quad e^z = e^{2\pi i} = 1,$$

so $\operatorname{Log}(e^z) = \operatorname{Log}(1) = 0.$

$$\rightsquigarrow \quad \operatorname{Log}(e^z) = 0 \neq z = 2\pi i.$$

$$\boxed{\log(z_1 z_2) = \log z_1 + \log z_2}$$

but (in general)

$$\boxed{\operatorname{Log}(z_1 z_2) \neq \operatorname{Log} z_1 + \operatorname{Log} z_2.}$$

$z_1 = -1$, $z_2 = -1$, $\operatorname{Log}(-1) = i\pi$, but
$\operatorname{Log}(z_1 z_2) = \operatorname{Log}(1) = 0 \neq \operatorname{Log} z_1 + \operatorname{Log} z_2 = 2i\pi.$

③ Power functions

Fix $c \in \mathbb{N}$. Then

$$c \log z = c(\ln|z| + i \arg z)$$
$$= \underbrace{c(\ln|z| + i \operatorname{Arg} z + 2n i \pi)}_{\text{multivalued}}, \quad n \in \mathbb{Z}.$$

$$\rightsquigarrow \quad e^{c \log z} = \underbrace{e^{c \ln|z|}}_{|z|^c} e^{ic \operatorname{Arg} z} \underbrace{e^{c 2n i \pi}}_{1}$$

$$\rightsquigarrow \quad e^{c \log z} = |z|^c e^{ic \operatorname{Arg} z} \quad \leftarrow \text{single valued}$$

Writing $z = re^{i\theta}$, $-\pi < \theta \le \pi$, we see that $z^c = r^c e^{in\theta} = |z|^c e^{ic \operatorname{Arg} z}$, so

$$\boxed{z^c = e^{c \log z}.}$$

Now let $c \in \mathbb{C}$, fixed.

Defⁿ: $\quad \underline{z^c = e^{c \log z}}, \quad z \neq 0.$

Comments: ① When $c \in \mathbb{Z}$ this agrees with the usual definition.

② When $c = \frac{1}{n}$, $n \in \mathbb{N}$, this defⁿ says that $z^{1/n}$ is the multivalued function giving all n n^{th} roots of z.

$c \in \mathbb{C}$, fixed.

<u>Def</u>

Def^n: The <u>Principal Value</u> of z^c is

$$\underline{\underline{P.V.}}\; z^c = e^{c\,\underline{\underline{\operatorname{Log} z}}}.$$

Principal Value　　　　　Principal Value of $\log z$

<u>Examples</u>:

① $i^i = e^{i \log i}$,

$$\log i = \underbrace{\ln |i|}_{0} + \underbrace{i\tfrac{\pi}{2} + 2n\pi i}_{(2n+\frac{1}{2})\pi i}, \quad n \in \mathbb{Z}.$$

$$\rightsquigarrow\; i^i = e^{i(2n+\frac{1}{2})\pi i} = e^{-(2n+\frac{1}{2})\pi}, \quad n \in \mathbb{Z}.$$

$$\text{P.V. } i^i = e^{-\frac{\pi}{2}}. \qquad \text{(take } n=0\text{)}$$

② $1^{1/\pi} = e^{\frac{1}{\pi}\log(1)} = e^{\frac{1}{\pi}(2n\pi i)}$

$$= e^{2ni}, \quad n \in \mathbb{Z}.$$

$$\text{P.V. } 1^{1/\pi} = 1.$$

Fix a branch of the logarithm:

analytic $\longrightarrow$ $\log z = \ln|z| + i \arg z, \quad \alpha < \arg z < \alpha + 2\pi.$

The corresponding branch of $z^c = e^{c \log z}$ is analytic and $\boxed{\frac{d}{dz}(z^c) = c z^{c-1}.}$

Here $z \neq 0$ & $\alpha < \arg z < \alpha + 2\pi$.

④ $\cos z$, $\sin z$, $\cosh z$, $\sinh z$

For $x \in \mathbb{R}$,
$$e^{ix} = \cos x + i \sin x,$$
$$e^{-ix} = \cos x - i \sin x.$$

Hence

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

Defn: For $z \in \mathbb{C}$, define the entire functions

$$\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2}}, \qquad \boxed{\sin z = \frac{e^{iz} - e^{-iz}}{2i}}.$$

$$(\cos z)' = \frac{ie^{iz} - ie^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z$$

$$(\sin z)' = \frac{ie^{iz} + ie^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

Defn: For $z \in \mathbb{C}$, define the entire functions

$$\boxed{\cosh z = \frac{e^{z} + e^{-z}}{2}}, \qquad \boxed{\sinh z = \frac{e^{z} - e^{-z}}{2}}.$$

$$(\cosh z)' = \sinh z, \qquad (\sinh z)' = \cosh z$$

$$\cosh(iz) = \cos z, \qquad \sinh(iz) = i \sin z.$$

(5) $\tan z$

The complex tangent function is defined by

$$\boxed{\tan z = \frac{\sin z}{\cos z}}$$

whenever $\cos z \neq 0$.

But when is $\cos z$ zero?

Note that $\cos z = \sin(z + \frac{\pi}{2})$ for all $z \in \mathbb{C}$ (use $e^{i\frac{\pi}{2}} = i$ in the def^n of $\sin z$).

$\leadsto$ Equivalent problem: when is $\sin z$ zero?

solve in 2 steps.

Problem 1: Find the real and imaginary parts of $\sin z$.

$$\leadsto \quad \sin z = \frac{e^{ix-y} - e^{-ix+y}}{2i}$$

$$= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i}$$

$$= \underbrace{\left(\frac{e^{-y} - e^{y}}{2i}\right)}_{i\sinh y} \cos x + \underbrace{\left(\frac{e^{-y} + e^{y}}{2}\right)}_{\cosh y} \sin x$$

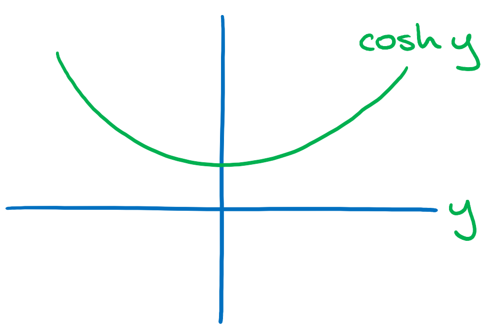
$$\rightsquigarrow \quad \boxed{\sin z = \sin x \cosh y + i \cos x \sinh y}$$

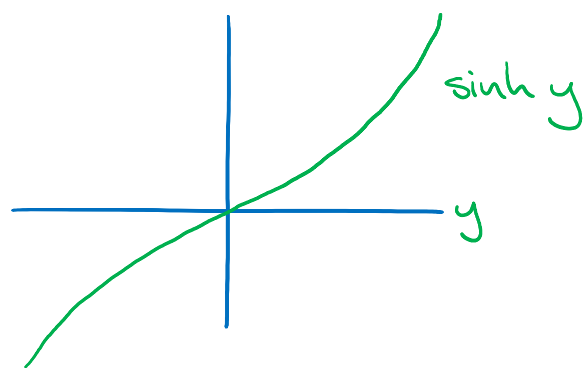
Problem 2: Find the zeros of $\sin z$.

$$\sin z = 0 \iff \begin{cases} \sin x \cosh y = 0 & (1) \\ \cos x \sinh y = 0 & (2) \end{cases}$$

(1): $\cosh y > 0$, so $\sin x \cosh y = 0$ gives $\sin x = 0$ and hence $x = k\pi$, $k \in \mathbb{Z}$.

(2): if $x = k\pi$, $k \in \mathbb{Z}$, then $\cos x = \pm 1$, so $\cos x \sinh y = 0$ gives $\sinh y = 0$. Hence $y = 0$.





$$\boxed{\sin z = 0 \iff z = k\pi, \ k \in \mathbb{Z}.}$$

$$\boxed{\cos z = 0 \iff z = k\pi + \frac{\pi}{2}, \ k \in \mathbb{Z}.}$$

→ the function $\tan z$ is analytic everywhere except at $z = k\pi + \frac{\pi}{2}$, $k \in \mathbb{Z}$.

Quotient rule gives $(\tan z)' = \sec^2 z$.

<u>Exercise:</u> Use the properties of the complex exponential to show:

① $\cos^2 z + \sin^2 z = 1$

② $\sec^2 z = \tan^2 z + 1$ (whenever $\cos z \neq 0$)

③ $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

④ $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

⑤ $\sin(2z) = 2 \sin z \cos z$