7. Analytic Functions

$S \subseteq \mathbb{C}$ open

"f is analytic on S"

Def^n: A function $f : S \to \mathbb{C}$ is analytic

$\Updownarrow$

$f'(z)$ exists for all $z \in S$.

Note: the term holomorphic is more standard than analytic here, but I will try to stay close to the book in my use of terms.

$S \subseteq \mathbb{C}$, not necessarily open,

$z_0 \in S$ an interior point

Def^n: $f : S \to \mathbb{C}$ is analytic at z_0

$\Updownarrow$

f is analytic in a neighborhood of z_0.

Note: sums, products, quotients, and compositions of analytic functions are analytic.

Def^n: $f(z)$ is an entire function

$\Updownarrow$

f is analytic on all of $\mathbb{C}$.

Examples:

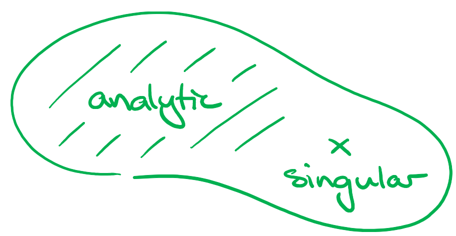
① polynomials in z $\leftarrow$ entire

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad a_j \in \mathbb{C}$$

② e^z entire

③ $f(z) = \dfrac{1}{z^2+1}$ analytic in $\mathbb{C} - \{i, -i\}$, not entire.

Def^n: If $f(z)$ is analytic in $0 < |z - z_0| < \varepsilon$ (deleted ε-neighborhood of z_0), but not analytic at z_0, then z_0 is called a singularity of f.



Examples:

① $f(z) = \dfrac{1}{z}$ has a singularity at $z = 0$.

② $f(z) = \dfrac{1}{z^2+1} = \dfrac{1}{(z-i)(z+i)}$

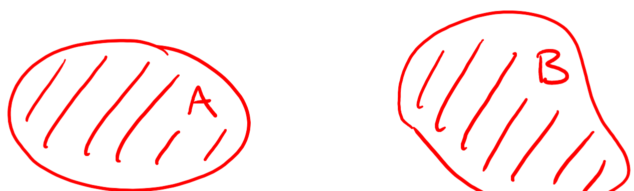
has singularities at $z = i$ and $z = -i$.

③ $f(z) = \dfrac{z^4+1}{(z+2)^2}$ has a singularity at $z = -2$.

Theorem: Let $f(z)$ be an analytic function on a domain $D \subseteq \mathbb{C}$ (i.e. an open and connected set). If $f'(z) = 0$ for all $z \in D$, then $f(z)$ is constant.

Remark: If D is not connected, then it is easy to see this result fails. Consider f given by:

$D = A \cup B$ ⟵ not connected!



$f(z) = 3$ on A $\qquad$ $f(z) = i$ on B

$f'(z) = 0$ for all $z \in D$, but f is not constant.

Proof of Theorem:

Since $f'(z) = u_x(x,y) + i v_x(x,y) = 0$ for all $z \in D$, we have $u_x = 0$ and $v_x = 0$ on D. By the C.-R. eqn.s we get $u_y = 0$ and $v_y = 0$.

⤳ the functions u & v are diff^{ble} on D and both have zero gradient on D.

$\nabla u = 0, \ \nabla v = 0$

D open & connected

⤳ u and v are constant. $\square$