4. Limits & Continuity

Let $f: S \to \mathbb{C}$ be a function, and $z_0 \in S$ an interior point of S.

Then $\lim_{z \to z_0} f(z) = L$ means

$f(z)$ approaches L as z approaches z_0.

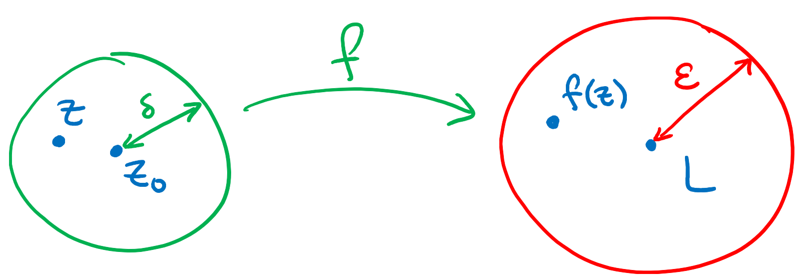
Formal def^n: $\lim_{z \to z_0} f(z) = L$ means:

for every $\varepsilon > 0$, we can find $\delta > 0$ (depending on ε) such that

if $0 < |z - z_0| < \delta$ then $|f(z) - L| < \varepsilon$.

$\rightsquigarrow$ Suppose we are given some $\varepsilon > 0$ (this $\varepsilon > 0$ is fixed, but could be arbitrarily small). If $\lim_{z \to z_0} f(z) = L$ then we can find a $\delta > 0$ small enough such that f maps the deleted δ-nbhd of z_0 into the ε-nbhd of L.

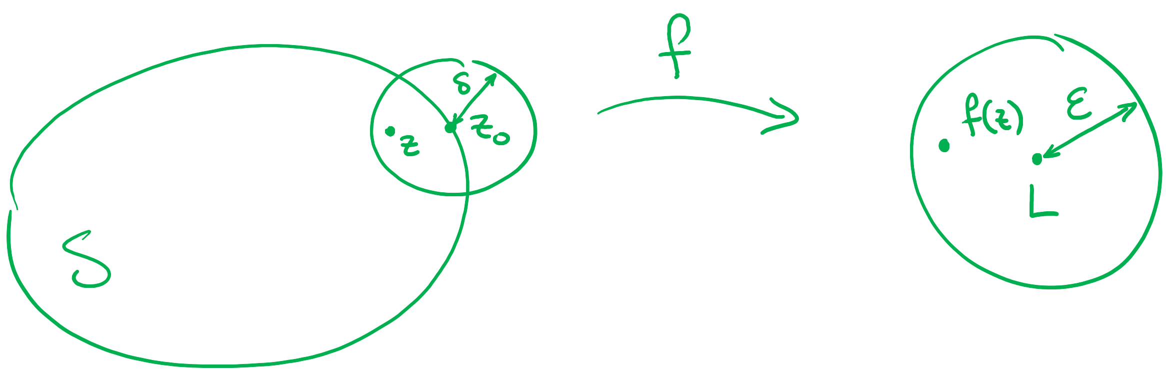
Note that when computing limits as $z \to z_0$ we do not care where f sends the point z_0, only where f sends the points z which are very close to z_0.



Remarks:

① Limits are unique!

② z_0 need not be in the interior of S, and could be any <u>accumulation point</u> of S.
In the def^n we would just replace "if $0<|z-z_0|<\delta$" with "if $z \in S$ and $0<|z-z_0|<\delta$" since being close to z_0 no longer forces z to be in S.



<u>Examples:</u>

① $f(z) = 2iz\,, \quad z_0 = 3$

$$\rightsquigarrow \quad \lim_{z\to z_0} f(z) = 6i = L$$

$f(z)-L = 2iz-6i = 2i(z-3)$

$|f(z)-L| = 2|z-3|.$

$\rightsquigarrow$ Given $\varepsilon>0$, take $\delta = \frac{\varepsilon}{2}$.

If $|z-3|<\delta=\frac{\varepsilon}{2}$, then $|f(z)-L|<2\delta=\varepsilon$.

$f : S \to \mathbb{C}$, z_0 an interior point of S.

Def^n: f is continuous at z_0

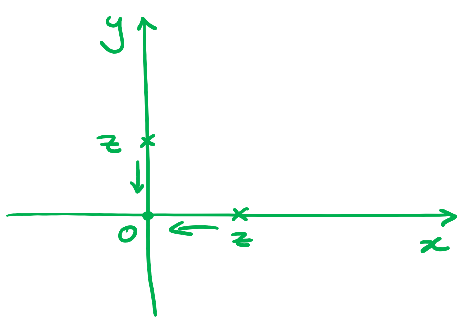
$$\Updownarrow$$

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Example:
$$f(z) = \begin{cases} \frac{z}{\bar{z}} & \text{if } z \in \mathbb{C} - \{0\} \\ 0 & \text{if } z = 0. \end{cases}$$

Is f continuous at 0?

Equivalently, is $\lim_{z \to 0} f(z) = 0$?



Consider approaching 0 along the two axes.

If $z = x = x + i0$, $\bar{z} = x - i0 = x$

so $f\left(\frac{z}{\bar{z}}\right) = \frac{x}{x} = 1$ $\quad (x \neq 0)$.

If $z = iy$, $\bar{z} = -iy$

so $f\left(\frac{z}{\bar{z}}\right) = \frac{iy}{-iy} = -1$ $\quad (y \neq 0)$.

$\rightsquigarrow$ f takes both the values $+1$ and -1 at points arbitrarily close to $z_0 = 0$.

$\rightsquigarrow$ $\lim_{z \to z_0} f(z)$ does not exist!

$\rightarrow$ f is not continuous at 0!

Theorems on limits (& continuity):

(No need to know proofs, but read book to understand.)

① $\lim_{z\to z_0}[f(z)+g(z)] = \lim_{z\to z_0} f(z) + \lim_{z\to z_0} g(z)$

② $\lim_{z\to z_0}[f(z)g(z)] = \left[\lim_{z\to z_0} f(z)\right]\left[\lim_{z\to z_0} g(z)\right]$

③ $\lim_{z\to z_0}\dfrac{f(z)}{g(z)} = \dfrac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)}$

provided $\lim_{z\to z_0} g(z) \neq 0$.

⇝ Sums & products of cts functions are cts, as are quotients when defined.

Examples:

① Every polynomial function

$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$,

$a_j \in \mathbb{C}$, is continuous at every $z_0 \in \mathbb{C}$.

② Rational functions $\dfrac{p(z)}{q(z)}$,

p, q polynomials,

are continuous if $q(z_0) \neq 0$.

③ $f(z) = \overline{z}$ is continuous at every $z_0 \in \mathbb{C}$.

(easy exercise)

Theorem: $f(z) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$.

$f(x,y)$ continuous at z_0

$$\Updownarrow$$

$u(x,y)$ & $v(x,y)$ continuous at (x_0, y_0).

Example: e^z is continuous (at every $z_0 \in \mathbb{C}$).

$$\rightsquigarrow \quad e^z = e^x e^{iy} = \underbrace{e^x \cos y}_{u(x,y)} + i \underbrace{e^x \sin y}_{v(x,y)} \quad \leftarrow \underline{\underline{\text{cts}}}$$

Composition: f, g functions

$$(f \circ g)(z) \overset{\text{def}^n}{=} f(g(z))$$

provided f is defined at $g(z)$.

Theorem: The composition of continuous functions is continuous.

Example: e^{3z^2-5} is continuous

(composition of $f(w) = e^w$, $g(z) = 3z^2 - 5$)

Limits involving ∞

The following "def^{n}s" are (essentially trivial) theorems in the book, which briefly explains how you can add one point ("∞") to the complex plane to get a sphere (the Riemann sphere). This is a very important idea, but we will not make a big deal about it right now.

Def^n:

① $$\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0,$$

② $$\lim_{z \to \infty} f(z) = w_0 \iff \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0,$$

③ $$\lim_{z \to \infty} f(z) = \infty \iff \lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0.$$

Examples:

① $f(z) = \dfrac{5}{z-i}$, $\displaystyle\lim_{z \to i} f(z) = ?$

$$\frac{1}{f(z)} = \frac{z-i}{5} \qquad \lim_{z \to i} \frac{1}{f(z)} = \frac{i-i}{5} = 0$$

Hence $\displaystyle\lim_{z \to i} \frac{5}{z-i} = \infty$.

② $$\lim_{z \to \infty} \frac{2z^2 - 5z + 2i}{4iz^2 + 7} = ? \qquad \boxed{\frac{-i}{2}}$$

$$f(z) = \frac{2z^2 - 5z + 2i}{4iz^2 + 7}, \qquad \lim_{z \to 0} f\left(\frac{1}{z}\right) = ?$$

$$f\left(\frac{1}{z}\right) = \frac{2z^{-2} - 5z^{-1} + 2i}{4iz^{-2} + 7} \times \frac{z^2}{z^2} = \frac{2 - 5z + 2iz^2}{4i + 7z^2}$$

$$\lim_{z\to 0} f\left(\frac{1}{z}\right) = \frac{2}{4i} = \frac{1}{2i} = \frac{-i}{2}.$$

3 $$\lim_{z\to\infty} \frac{z^3 - i}{z^2+5} = ? \quad \boxed{\infty}$$

$$f(z) = \frac{z^3-i}{z^2+5}, \quad \text{need } \lim_{z\to 0} f\left(\frac{1}{z}\right)$$

$$f\left(\frac{1}{z}\right) = \frac{z^{-3} - i}{z^{-2} + 5} = \frac{1 - iz^3}{z + 5z^3} \leftarrow$$ looks like $\frac{1}{z}$ as $z \to 0$

$$\frac{1}{f\left(\frac{1}{z}\right)} = \frac{z - 5z^3}{1 - iz^3} \longrightarrow 0 \text{ as } z \to 0.$$

Hence $\lim_{z\to\infty} f(z) = \infty$.