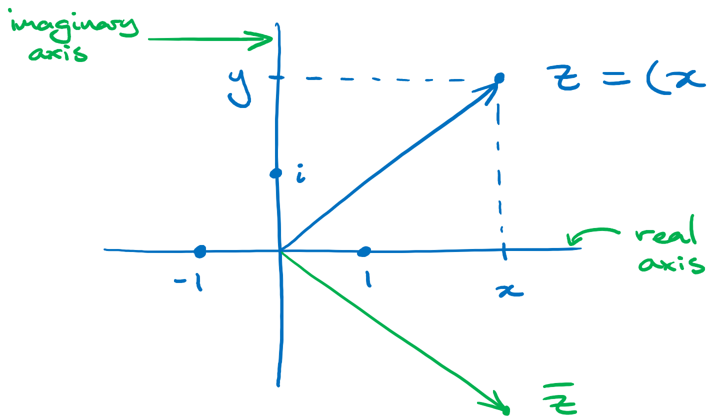
1. Review of Complex Numbers



vector notation

$$i = (0,1) \, , \qquad z = x + iy$$

The complex numbers form a "field":

i.e. the usual rules of addition and multiplication that work for real numbers also work for complex numbers.

addition — vector addition

multiplication $\quad i^2 = -1$

$$\rightsquigarrow \quad (a+ib)(c+id) = (ac-bd) + i(bc+ad)$$

commutativity: $\quad z_1 + z_2 = z_2 + z_1$

$$z_1 z_2 = z_2 z_1$$

distributivity: $\quad z_1(z_2+z_3) = z_1 z_2 + z_1 z_3$

associativity $\quad \ldots \quad$ ✓

subtraction ✓

! division: if $z \neq 0$, $z = a+ib$, then

$$z^{-1} = \frac{1}{a+ib} = \frac{a}{a^2+b^2} - i\,\frac{b}{a^2+b^2}\,.$$

multiply by $\dfrac{a-ib}{a-ib}$

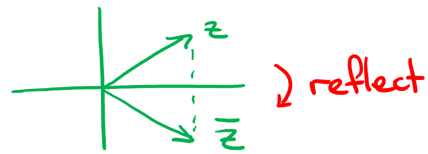
$\mathbb{C}$ = set of complex numbers

$\mathbb{R}$ = set of real numbers

$$\mathbb{R} \subset \mathbb{C}$$

$$x \longmapsto (x,0) = x + i0$$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$ integers.



$z = x + iy \in \mathbb{C}$

complex conjugate: $\bar{z} = x - iy$

properties: $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$$x = \operatorname{Re} z = \frac{z + \bar{z}}{2}$$

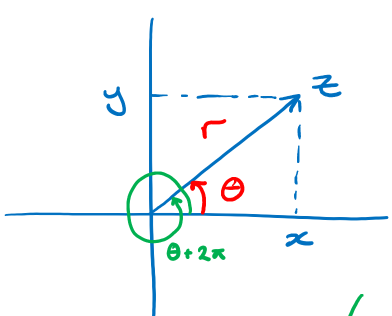
$$y = \operatorname{Im} z = \frac{z - \bar{z}}{2i}$$

Polar/exponential form

$z \neq 0, \; z \in \mathbb{C}$

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

"argument"

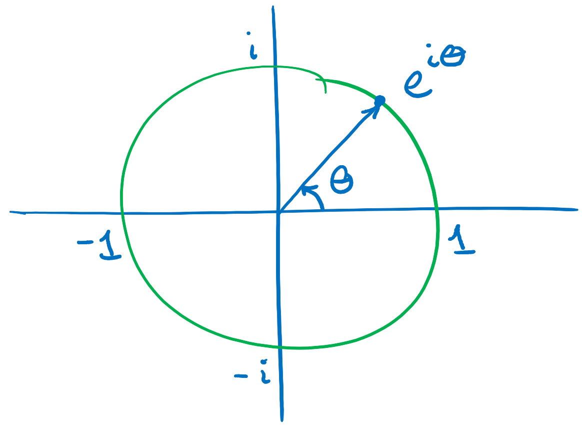


trig: $x = r\cos\theta$

$y = r\sin\theta$

(can add any integer multiple of 2π to θ)

Euler formula: $\underbrace{e^{i\theta} \overset{\text{def}^n}{=} \cos\theta + i\sin\theta}$



$\rightsquigarrow$ for $z \neq 0$, $\quad z = re^{i\theta}$

$r > 0$,

$\theta = \arg z$ defined up to $2k\pi$, $k \in \mathbb{Z}$.

Def^n/Convention:

$z \neq 0$ always has a unique argument θ such that $-\pi < \theta \leq \pi$.

$\hookrightarrow$ call this the principal argument $\operatorname{Arg} z$

$$\boxed{-\pi < \operatorname{Arg} z \leq \pi}$$

$\operatorname{Arg}(1) = 0$, $\quad \arg(1) = 0 + 2k\pi$, $k \in \mathbb{Z}$

$\operatorname{Arg}(i) = \frac{\pi}{2}$, $\quad \arg(i) = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$

$\operatorname{Arg}(-1) = \pi$, $\quad \arg(-1) = \pi + 2k\pi$, $k \in \mathbb{Z}$

$\operatorname{Arg}(1+i) = \frac{\pi}{4}$, $\quad \arg(1+i) = \frac{\pi}{4} + 2k\pi$, $k \in \mathbb{Z}$.

Modulus

$$z = x + iy \in \mathbb{C}$$

$$|z| \stackrel{\text{def}^n}{=} \sqrt{x^2 + y^2}$$

modulus = length

Note:

- if $z \neq 0$, $z = x + iy = re^{i\theta}$,
  then $|z| = r$.
- note that $|e^{i\theta}| = (\cos^2\theta + \sin^2\theta)^{1/2} = 1$
- if $z = x + iy$,

$$z\bar{z} = (x+iy)(x-iy) = x^2 + y^2$$

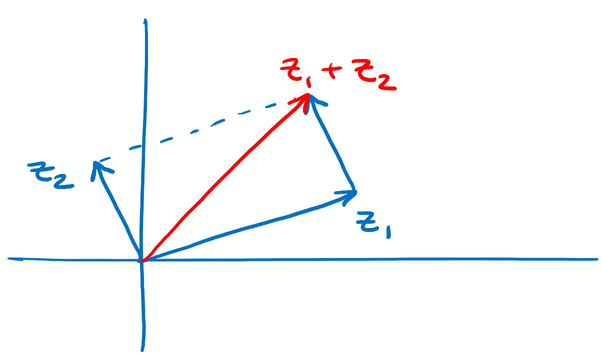
$$\rightsquigarrow \quad z\bar{z} = |z|^2$$

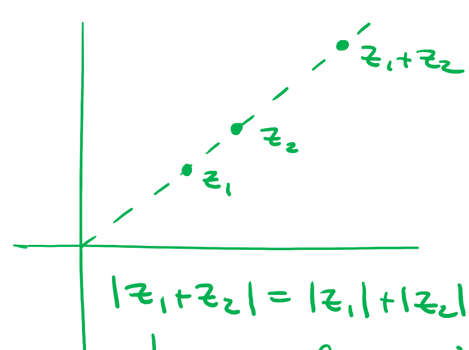
so $\quad |z| = \sqrt{z\bar{z}}$

Triangle Inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

"the shortest path between two points is a straight line"





$|z_1 + z_2| = |z_1| + |z_2|$ when z_1 & z_2 point in the same direction.

To make $|z_1+z_2|$ "as large as possible" we should make z_1 and z_2 point in the same direction.

Q: How do we make $|z_1+z_2|$ "as small as possible"?

A: Make z_1 and z_2 point in the opposite direction.

Exercise: use the triangle inequality to prove

$$\big|\,|z_1|-|z_2|\,\big| \le |z_1+z_2|.$$

we get $=$ here when z_1 & z_2 point in opposite directions

Multiplying Complex Numbers in Polar Form

$z_1, z_2 \neq 0, \qquad z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}$

$$\boxed{z_1 z_2 = \underbrace{r_1 r_2}_{\text{multiply moduli}}\, e^{i\overbrace{(\theta_1+\theta_2)}^{\text{add arguments}}}}$$

Proof: use $e^{i\theta} = \cos\theta + i\sin\theta$ and trig. formulae for sums of angles.

Note: If θ_1 & θ_2 are principal arguments (i.e. $-\pi < \theta_1, \theta_2 \le \pi$) then $\theta_1+\theta_2$ may not be principal. We may write

$\arg(z_1z_2) \overset{\checkmark}{=} \arg(z_1)+\arg(z_2)$

but not

$\operatorname{Arg}(z_1z_2) \neq \operatorname{Arg}(z_1)+\operatorname{Arg}(z_2)$.

$$\leadsto \quad |z_1 z_2| = |z_1|\,|z_2|$$

and

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

Powers: If $z = re^{i\theta}$, then $z^n = r^n e^{in\theta}$.

In particular,

$$(e^{i\theta})^n = e^{in\theta}.$$ <u>de Moivre's formula</u>

<u>n^{th} roots</u>

Problem: Given $z_0 \in \mathbb{C}$ and $n \in \{1,2,3,4,\ldots\}$ solve

$$z^n = z_0.$$

(We may as well assume $z_0 \neq 0$, since if $z_0 = 0$ the only solution is $z = 0$.)

<u>warm up</u>

Square roots $(n=2)$: Assume $z_0 \neq 0$.

Let $\theta_0 = \operatorname{Arg} z_0$, so $\arg z_0 = \theta_0 + 2k\pi$, $k \in \mathbb{Z}$.

Let $r_0 = |z_0|$.

$$\rightsquigarrow \quad z_0 = r_0 e^{i(\theta_0 + 2k\pi)}$$

Substitute $z = re^{i\theta}$ into $z^2 = z_0$

$$\rightsquigarrow \quad r^2 e^{i2\theta} = r_0 e^{i(\theta_0 + 2k\pi)}$$

$\longrightarrow \quad r^2 = r_0 \quad \longrightarrow \quad r = r_0^{1/2} = \sqrt{r_0}$

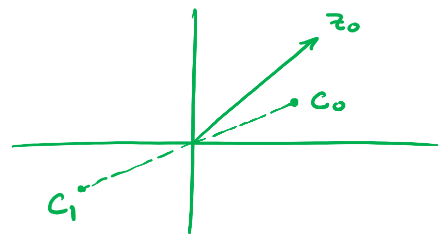
$\longrightarrow \quad 2\theta = \theta_0 + 2k\pi \quad \longrightarrow \quad \theta = \frac{\theta_0}{2} + k\pi, \; k \in \mathbb{Z}$

$$\rightsquigarrow \text{ solutions } \quad z = \sqrt{r_0}\, \underbrace{e^{i\left(\frac{\theta_0}{2} + k\pi\right)}}_{= e^{i\frac{\theta_0}{2}} \underbrace{e^{ik\pi}}_{\substack{+1 \text{ if } k \text{ is even} \\ -1 \text{ if } k \text{ is odd}}}}, \quad \underline{k = 0, 1}.$$

label the solutions:

"principal square root" $\longrightarrow$ $c_0 = \sqrt{r_0}\, e^{i\frac{\Theta_0}{2}}$ $(k=0)$

$c_1 = \sqrt{r_0}\, e^{i\frac{\Theta_0}{2}} e^{i\pi} = -c_0$ $(k=1)$.



General case $(n \geq 2)$: Assume $z_0 \neq 0$.

$\rightsquigarrow$ $z_0 = r_0\, e^{i(\Theta_0 + 2k\pi)}$, $\Theta_0 = \operatorname{Arg} z_0$.

$\underline{z^n = z_0} \xrightarrow{z = re^{i\theta}} r^n e^{in\theta} = r_0 e^{i(\Theta_0 + 2k\pi)}$

want to solve for z

$\rightsquigarrow$ $r = \sqrt[n]{r_0}$ (r is unique)

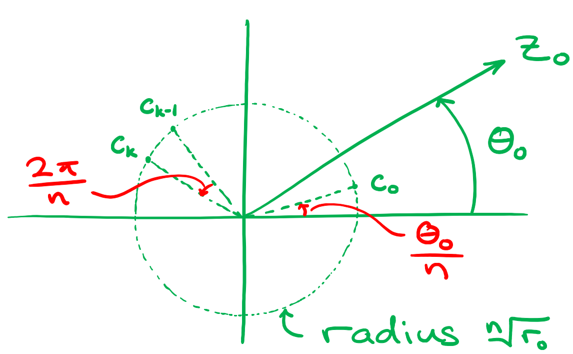
$$\underbrace{\theta = \frac{\Theta_0}{n} + \frac{2k\pi}{n}, \quad k \in \mathbb{Z}}_{}$$

Since $z = re^{i\theta}$ we only need $k = 0, 1, \ldots, n-1$.

Solutions are $c_k = \sqrt[n]{r_0}\, e^{i\left(\frac{\Theta_0}{n} + \frac{2k\pi}{n}\right)}$,

$k = 0, 1, \ldots, n-1$.

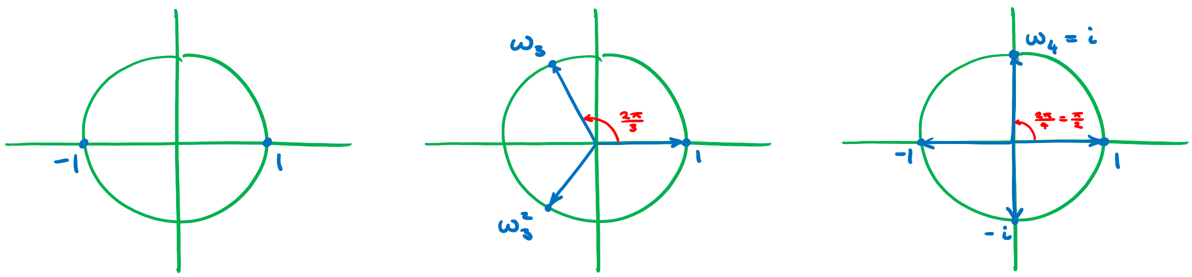
c_0 is called the principal n^{th} root.



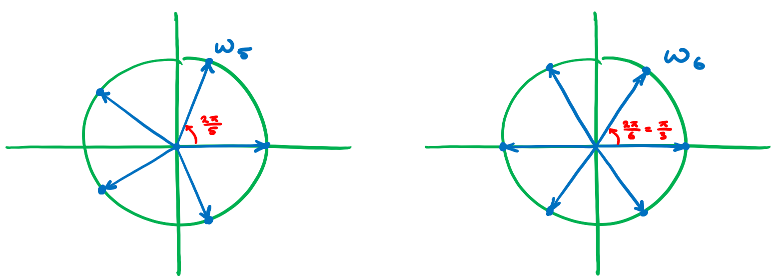
n^{th} roots of unity $\rightarrow$ Solve $z^n = 1$.

Let $\omega_n = e^{i\frac{2\pi}{n}}$. The solutions of

$z^n = 1$ are $\underset{c_0}{\overset{\shortparallel}{1}}, \underset{c_1}{\overset{\shortparallel}{\omega_n}}, \underset{c_2}{\overset{\shortparallel}{\omega_n^2}}, \ldots, \underset{c_{n-1}}{\overset{\shortparallel}{\omega_n^{n-1}}}$. $\leftarrow$ these form the vertices of a regular n-polygon.



$n = 2$ $\quad$ $n = 3$ $\quad$ $n = 4$



$n = 5$ $\quad$ $n = 6$

Note:

When we solve $z^n = z_0$ for z with $z_0 = r_0 e^{i\theta_0} \neq 0$ we get the solutions

$$c_k = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n} + \frac{2\pi k}{n}\right)}, \quad k = 0, 1, \ldots, n-1.$$

These can be written as

$$c_k = \sqrt[n]{r_0}\, e^{i\frac{\theta_0}{n}}\, \omega_n^k, \quad k = 0, 1, \ldots, n-1.$$