1. Let P and Q be two statements. Prove by using truth tables that the following statements are equivalent.

(a) \( P \lor Q \)

(b) \( (\neg Q) \Rightarrow P \)

\[
\begin{array}{c|c|c|c}
P & Q & P \lor Q & \neg Q \Rightarrow P \\
\hline
T & T & T & T \\
T & F & T & T \\
F & T & T & T \\
F & F & F & F \\
\end{array}
\]

Note that \( P \lor Q \) and \( \neg Q \Rightarrow P \) have the same true table. So they are equivalent.
2. Let $a_n$ be a sequence defined by

$$a_1 = 1, a_2 = 1, a_3 = 1; \quad a_n = 2a_{n-2} + 4a_{n-3} \text{ for } n \geq 4;$$

(a). Compute the values of $a_4, a_5$.

$$a_4 = 2a_2 + 4a_1 = 6$$

$$a_5 = 2a_3 + 4a_2 = 6$$

(b). Prove that $a_n \leq 2^n$ for all $n \geq 1$. At the beginning of your proof, make sure to indicate which induction you use.

**Proof:** Using strong induction.

Note that it is true for $n = 1, 2, 3$.

Suppose for $n = 1, 2, \ldots, k$ with $k \geq 3$, we have $a_n \leq 2^n$.

Then $a_{k+1} = 2a_{k-1} + 4a_{k-2} \leq 2 \cdot 2^{k-1} + 4 \cdot 2^{k-2}$

by strong induction

$$= 2^k + 2^k = 2^{k+1}$$

So we prove it by strong induction.
(10) 3. Prove for every integer \( n \geq 1 \), \( 7^{n+1} + 8^{2n-1} \) is divisible by 57.

\[
\text{Pf. For } n=1, \quad 7^2 + 8^1 = 57 \mid 57 \quad \checkmark.
\]

Suppose it’s true for some \( n \geq 1 \),

Note that
\[
7^{n+2} + 8^{2n+1} = 7 \cdot 7^{n+1} + 8^2 \cdot 8^{2n-1}
\]

\[
= 7 \cdot 7^{n+1} + (57+7) \cdot 8^{2n-1}
\]

\[
= 7 \cdot (7^{n+1} + 8^{2n-1}) + 57 \cdot 8^{2n-1}
\]

By induction \( 57 \mid 7^{n+1} + 8^{2n-1} \)

So \( 57 \mid 7^{n+2} + 8^{2n+1} \)

Proved by induction
4. Prove that there do not exist integers $m$ and $n$ such that \( \frac{9m+3}{2n+4} = 9 \).

\[ \begin{align*}
\text{Pf: Assume that } & \exists \ m, n \in \mathbb{Z} \ s.t. \quad \frac{9m+3}{2n+4} = 9. \\
\implies & \quad 9m+3 = 18n+36 \\
\implies & \quad 9m-18n = 33 \\
\implies & \quad 3m-6n = 11 \\
\implies & \quad 3(m-n) = 11 \implies 3 \text{ divides } 11.
\end{align*} \]

This is a contradiction.
Let \( X = \{1, 2, 3, 4\}, Y = \{1, 9\} \). Find \( X - Y, Y - X \) and \( \mathcal{P}(X - Y) \).

\[
X - Y = \{2, 3, 4\} \\
Y - X = \{9\} \\
\mathcal{P}(X - Y) = \{\emptyset, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}\}.
\]

Select all the true statements from the following ones. You don’t need to prove or justify. For example, if you think (1) and (2) are the only true ones, then you just need to write “The true statements are: (1), (2)”.

1. For any two sets \( X \) and \( Y \), we have \( \mathcal{P}(X - Y) = \mathcal{P}(X) - \mathcal{P}(Y) \).
2. Let \( A, B \) be two sets. If \( A - B = B - A \), then \( A = B \).
3. Let \( n \) be an integer. ‘4 divides \( n \)’ is necessary for ‘2 divides \( n \)’.
4. Let \( x \) be a real number. If \( x^4 + 2 = 0 \), then \( x < 0 \).
5. For any two sets \( X \) and \( Y \), we have \( \mathcal{P}(X \cup Y) = \mathcal{P}(X) \cup \mathcal{P}(Y) \).
6. ‘\( n=1 \)’ is sufficient for ‘\( n^2 + 2n - 3 = 0 \)’.

\( F: \) RHS no \( \emptyset \).

\( T: \) If \( x \in A \) but \( x \notin B \), then \( x \in A - B \) but \( x \notin B - A \).

\( F: \) \( 2 \mid 6 \) but \( 4 \nmid 6 \).

\( T: \) \( x^4 + 2 = 0 \) is false for \( \forall x \in \mathbb{R} \).

\( F: \) \( X = \{1, 2\}, Y = \{3, 4\}, \) but \( \{1, 3\} \notin \mathcal{P}(X) \cup \mathcal{P}(Y) \).
6. Give an example of three sets $X, Y, Z$ such that the following conditions are all satisfied.

1. $X - Y = X - Z$;
2. $Y - Z = \{1\}$.
3. $X$ has one more element than $X - Y$.
4. $\mathcal{P}(Y), \mathcal{P}(Z), \mathcal{P}(X - Y)$ all have 4 elements.

\[ Y, Z, X - Y \text{ have 2 elements.} \]

\[ \text{e.g.: } Y = \{1, 3\}, \quad \text{check: } 1 X - Y = \{3, 4\} = X - Z. \]

\[ Z = \{0, 2\}, \quad \text{② } Y - Z = \{1\}. \]

\[ X = \{0, 3, 4\}, \quad \text{③ } \checkmark \]

\[ \text{④ } \checkmark \]

The exam ends here.