PROPER HOLOMORPHIC MAPS FROM THE UNIT DISK TO SOME UNIT BALL

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Abstract. We study proper rational maps from the unit disk to balls in higher dimensions. After gathering some known results, we study the moduli space of unitary equivalence classes of polynomial proper maps from the disk to a ball, and we establish a normal form for these equivalence classes. We also prove that all rational proper maps from the disk to a ball are homotopic in target dimension at least 2.

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1. Introduction

There is a vast literature on proper holomorphic mappings between balls in possibly different dimensional complex Euclidean spaces. See for example [D1], [D2], [F], [H], [HJ], [JZ] and their references. See also [BEH], [EHZ], [L] and their references for related work when the target hypersurface is a hyperquadric rather than a sphere. Many of these results assume the domain dimension is at least two. In this paper we gather together various facts when the domain is one-dimensional and prove several new results in this setting. In particular we compute the moduli space of unitary equivalence classes of polynomial maps taking the unit circle to some unit sphere and we find a normal form. See Theorem 4.1. To do so we analyze what we call upper-trace identities. We also give a new result on homotopy classes in target dimension at least 2.

Section 2 on singularities gathers together some known results when the source dimension is 1. Section 3 on equivalence relations helps clarify three notions arising in the study of proper mappings between balls: spherical equivalence, unitary equivalence, and homotopy equivalence. Section 4 includes the new results on unitary equivalence for polynomial maps sending the unit circle in some unit sphere. Section 5 includes a proof that, when the target dimension is at least 2, there is but one homotopy equivalence class for rational proper maps from the disk to a ball.

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2. singularities

In this section we consider singularities for proper holomorphic maps from the unit disc Δ in $\mathbb{C}$ to the unit ball $\mathbb{B}_N$ in $\mathbb{C}^N$ for $N \geq 2$. The case when the target is one-dimensional is classical; such maps are precisely the finite Blaschke products. Such maps are of course rational and extend holomorphically past the unit circle. It is known ([CS]) that rational proper holomorphic maps from $\mathbb{B}_n$ to some $\mathbb{B}_N$ also extend holomorphically past the sphere. Furthermore, if $n \geq 2$, and a proper holomorphic map $f : \mathbb{B}_n \to \mathbb{B}_N$ extends past the sphere, then it is rational ([F]). It is worth observing the following result; when the domain is the unit disk and the target ball is higher dimensional, there are proper maps that extend past the circle but that are not rational.

Proposition 2.1. *For each $N \geq 2$, there is a non-algebraic holomorphic proper map from the open unit disk Δ to $\mathbb{B}_N$ that extends holomorphically to a neighborhood of the closed unit disk.*

Proof. Write $f_1(z) = e^{z-2}$. Then $|f_1| = e^{x-2} < 1$ on the unit circle. Let $\phi(x)$ be the real analytic function in a neighborhood of $\overline{\Delta}$ such that $e^{2\phi(x)} + e^{2(x-2)} = 1$ on the unit circle. Let $u(z)$ be the harmonic function with boundary value ϕ. Since ϕ is real analytic in a neighborhood D of $\overline{\Delta}$, also u is real-analytic there. Let v be the harmonic conjugate of u in D and put $h = u + iv$. Then h and thus $f_2 = e^h$ is holomorphic in D. Put $F = (f_1, f_2)$. On the circle,

$$||F||^2 = |f_1|^2 + |f_2|^2 = e^{2(x-2)} + e^{2\phi} = 1$$

and hence F maps the unit circle to the unit sphere. Also, F is not algebraic. □

We next observe the following simple fact about singularities of rational proper maps from Δ to $\mathbb{B}_N$.

Proposition 2.2. *Let $F = \frac{(p_1, \ldots, p_N)}{q}$ be a holomorphic rational proper map from Δ to $\mathbb{B}_N$. Then F extends holomorphically across the unit circle.*

Proof. We may assume that F is reduced to lowest terms. Suppose $q(z_0) = 0$ for some z_0 on the circle. Since q is a polynomial, it is divisible by $(z - z_0)$. Since F is proper, $||F(z)||^2$ tends to 1 as z tends to z_0. Therefore $||p(z)||^2$ tends to $|q(z_0)|^2 = 0$ as z tends to z_0. Hence each component of p is also divisible by $(z-z_0)$, and F is not reduced to lowest terms. Thus q does not vanish on the circle and the conclusion follows. □

Let $z \in \mathbb{C}^n$. We use the term **rational sphere map** for a rational function $z \mapsto \frac{p(z)}{q(z)}$ that maps the unit sphere in the source to the unit sphere in the target. We assume without loss of generality that p and q have no common factor and, when $n \geq 2$, that $q(0) = 1$. Each rational proper map between balls defines a rational sphere map; for $n \geq 2$ the only other examples are constants. When $n = 1$, however, rational sphere maps also include maps that have singularities in the unit disk. In particular the denominator can have a power of z as a factor. The following elementary result indicates that the denominator can vanish at points inside the disk; the a_k in this lemma can be anywhere except on the circle itself.

Lemma 2.1. *Suppose $\frac{p}{q}$ is a rational function mapping the unit circle to the unit sphere in some $\mathbb{C}^N$, and $\frac{p}{q}$ is reduced to lowest terms. Then the denominator q can be written*

$$q(z) = cz^m \prod_{k=1}^{K} (1 - \overline{a}_k z), \tag{1}$$

where c is a constant, $m \geq 0$, and $|a_k| \neq 1$ for each k.

3. Equivalence relations for proper mappings

We introduce various equivalence relations for proper holomorphic mappings $f : \mathbb{B}_n \to \mathbb{B}_N$ between balls. We first mention a convenient notation: we often write maps of the form $(f, 0)$ as $f \oplus 0$. Here we have not specified the number of zero components.

Because the automorphism group of the unit ball (in each dimension) is transitive (and hence large), it is natural to consider *spherical equivalence* of proper maps.

Definition 3.1. Proper mappings f, g from $\mathbb{B}_n$ to $\mathbb{B}_N$ are *spherically equivalent* if there are automorphisms ψ of the target ball and χ of the domain ball such that $f = \psi \circ g \circ \chi$. Proper maps are *unitarily equivalent* if both of these automorphisms can be chosen to be unitary maps.

The unitary group $\mathbf{U}(n)$ is a subgroup of the automorphism group. It is natural in some problems to consider unitary equivalence rather than spherical equivalence. Homotopy equivalence [DL1] also plays a crucial role. See also [DL2].

Definition 3.2. Let $f : \mathbb{B}_n \to \mathbb{B}_{N_f}$ and $g : \mathbb{B}_n \to \mathbb{B}_{N_g}$ be proper holomorphic maps. We say f and g are *homotopic in target dimension* N if there is a one-parameter family of proper maps $H_t : \mathbb{B}_n \to \mathbb{B}_N$ such that $H_0 = f \oplus 0$ and $H_1 = g \oplus 0$. We assume the map $(z, t) \to H_t(z)$ is continuous.

Example 3.1. Spherical equivalence implies homotopy equivalence. See [DL1] for the simple proof, which relies on the explicit form of the automorphisms and the path-connectedness of the unitary group. In one dimension we deform the map $e^{i\theta}\frac{z-a}{1-\overline{a}z}$ to the identity by replacing a with $(1-t)a$ and θ with $(1-t)\theta$. In Theorem 5.1 we prove that a holomorphic rational proper map from the disk to a ball and of degree one is homotopic to $z \oplus 0$.

We can also consider homotopy equivalence for rational sphere maps. For $n \geq 2$, we are including constant maps as well as proper maps between balls. When $n = 1$, by Lemma 2.1, we are also including functions with poles inside the unit disk. For rational sphere maps that are holomorphic on the ball, thus in particular when $n \geq 2$, the Taylor coefficients of a homotopy depend continuously on t. When $n = 1$, rational sphere maps with a pole at 0 do not have a Taylor expansion at 0.

The next lemma indicates how homotopy depends upon the target dimension.

Lemma 3.1. *If $f, g : \mathbb{B}_n \to \mathbb{B}_N$ are proper maps, then $f \oplus 0$ and $g \oplus 0$ are homotopic in target dimension $N + n$. If $n \geq 2$ and f, g are rational sphere maps, then $f \oplus 0$ and $g \oplus 0$ are homotopic in dimension $n + 1$.*

Proof. Let z denote the identity map. Note that $f \oplus 0$ is homotopic to $0 \oplus z$ in target dimension $N + n$ via the homotopy $(\sqrt{1-t^2}\ f, tz)$. The same applies to $g \oplus 0$. Since homotopy equivalence is an equivalence relation, $f \oplus 0$ and $g \oplus 0$ are homotopic in target dimension $N + n$.

If we regard f and g as rational sphere maps, thus allowing constant maps, the same argument works with z replaced by the constant function 1. Thus $f \oplus 0$ and $g \oplus 0$ are homotopic in target dimension $N + 1$. □

Remark 3.1. When the source dimension exceeds 1, the degree of a polynomial or rational proper map is **not** a homotopy invariant. See [DL1] for an explicit one-parameter family H_t of polynomial maps from $\mathbb{B}_2$ to $\mathbb{B}_5$ such that the embedding dimension of each H_t is 5 and yet the degree is not constant in t.

4. UNITARY EQUIVALENCE

We now assume that the source dimension is 1. Theorem 4.1 provides a complete analysis of the polynomial case. Consider a polynomial $f : \mathbb{C} \to \mathbb{C}^N$ that maps the circle to the unit sphere. Thus $||f(z)||^2 = 1$ on the circle. This condition leads to a system of linear equations for the inner products of the coefficient vectors. Since U is unitary if and only if U preserves all inner products, this system of equations takes into account unitary equivalence in the target space. The identities (3) and (4) below play a major role in this paper. They are special cases of the upper-trace identities from Definition 4.2.

Let $f : \mathbb{C} \to \mathbb{C}^N$ be a polynomial of degree d. If the coefficient vectors are linearly independent, then $N \geq d + 1$. Without loss of generality we will assume that $N = d + 1$ when we have a polynomial of degree d.

Proposition 4.1. *Let $f : \mathbb{C} \to \mathbb{C}^N$ be a polynomial of degree d. For $A_j \in \mathbb{C}^N$ put*

$$f(z) = \sum_{j=0}^{d} A_j z^j. \tag{2}$$

Then f maps the unit circle to the unit sphere if and only if the inner products $B_{jk} = \langle A_j, A_k \rangle$ of the coefficient vectors A_j satisfy the following linear system:

$$\sum_{j=0}^{d} B_{jj} = 1 \tag{3}$$

$$\text{For } l \neq 0, \quad \sum_{k=0}^{d-l} B_{(k+l)k} = 0. \tag{4}$$

Proof. On the circle we have the condition

$$1 = ||f(z)||^2 = ||\sum_{j=0}^{d} A_j z^j||^2 = \sum_{j,k} \langle A_j, A_k \rangle z^{j-k}. \tag{5}$$

Replace z by $e^{i\theta}$ in (5) and equate Fourier coefficients. For each l we get

$$1 = \sum_{k} B_{(k+l)k} e^{il\theta}.$$

Equation (3) follows because the constant term equals 1. Equation (4) follows from equating the coefficient of $e^{il\theta}$ to 0. □

Note that a matrix of inner products is non-negative definite, and hence all of its principal minor determinants are non-negative. We write $B(f)$ for the matrix of inner products of a polynomial sphere map f.

Definition 4.1. Let $B = (B_{jk})$ and $C = (C_{jk})$ be square Hermitian matrices of the same size. We define an equivalence relation $\sim$ by $B \sim C$ if there is a θ such that

$$B_{j(j+m)} = e^{im\theta} C_{j(j+m)}$$

for all j and m. We say simply that B and C are *-equivalent.

Recall that f and g are unitarily equivalent if $f = U \circ g \circ u$ for some $u \in \mathbf{U}(1)$ and some $U \in \mathbf{U}(d + 1)$.

Corollary 4.1. *Let f and g be proper polynomial maps from Δ to $\mathbb{B}_{d+1}$ of degree d. Then f and g are unitarily equivalent if and only if $B(f)$ is *-equivalent to $B(g)$.*

Corollary 4.2. *The moduli space of unitarily equivalent polynomial sphere maps of degree d from the circle to S^{2d+1} is the quotient space $W/\sim$, where W is the collection of non-negative $(d+1)$-by-$(d+1)$ Hermitian matrices satisfying (3) and (4) above, and $\sim$ is defined as in Definition 4.1.*

There are $\frac{d(d+1)}{2}$ parameters, of which d are real. If all the off-diagonal elements of B are non-negative real numbers, then we can go no further using the group $\mathbf{U}(1)$. Otherwise we can make a chosen non-zero off-diagonal element positive via a rotation in the source, and thus we can make an additional parameter real. The moduli space is a semi-algebraic real subset of $\mathbb{C}^{(d+1)^2}$.

We want to give normal forms for the unitary equivalence classes. For small target dimension and/or low degree it is fairly easy to solve the linear system from Proposition 4.1. After stating the general result, we list the normal forms explicitly for $d \leq 3$.

The idea is simple. We regard the coefficients as vectors in the target space and select an orthonormal basis in which their matrix representation is as simple as possible. Proposition 4.1 shows that the condition of mapping the circle to the sphere depends only on the inner products of these vectors, and hence this matrix depends only upon unitary transformations in the target.

Theorem 4.1. *Let $f(z) = \sum_{j=0}^{d} A_j z^j$ be a polynomial sphere map of degree d. Without loss of generality assume that $f : \mathbb{C} \to \mathbb{C}^{d+1}$. For each d, there is an orthonormal basis $e_0, ..., e_d$ of the target $\mathbb{C}^{d+1}$ in which*

$$A_j = \sum a_{jk} e_k$$

and these vectors (and hence f) are in the following partial normal form with respect to $\mathbf{U}(d+1)$:

- $A_0 = ||A_0||e_0$ and $A_d = ||A_d||e_d$.
- $a_{(d-1)0} = -\alpha||A_d||$ for some complex parameter α.
- $a_{1d} = \overline{\alpha}||A_0||$.
- The other entries in the first and last row are complex parameters.
- The **inner matrix** a_{jk} for $1 \leq j, k \leq d-1$ is upper-triangular.
- The first $d-2$ diagonal elements of the inner matrix are non-negative real parameters; the last diagonal entry is real, non-negative and determined.

Remark 4.1. There is no unique way to put the inner matrix into upper-triangular form, and hence our partial normal form is not unique. For small d, in the next propositions we provide the extra information needed for uniqueness.

Let $\mathcal{A}$ denote the matrix with column vectors A_j in this basis. Then $\mathcal{A}$ satisfies all the identities in (4). In particular, trace$(\mathcal{A}^*\mathcal{A}) = 1$. Hence the parameters in Theorem 4.1 are constrained by various inequalities. For example, Proposition 4.4 below gives precise ranges of values for these parameters in case f is degree 2.

Proposition 4.2. *Let f be as in Theorem 4.1. Consider the $(d+1)$ by $(d+1)$ normal form matrix $\mathcal{A}$ from Theorem 4.1. For $0 \leq d \leq 3$ these matrices have the following explicit forms:*

- *When $d = 0$, the normal form is $(1) = (||A_0||)$. (no parameters)*

- *When $d = 1$, the normal form is*

$$\begin{pmatrix} ||A_0|| & 0 \\ 0 & ||A_1|| \end{pmatrix}$$

where $||A_0||^2 + ||A_1||^2 = 1$. (one real parameter, either $||A_0||$ or $||A_1||$.)

- *When $d = 2$ the normal form is*

$$\begin{pmatrix} ||A_0|| & -\alpha||A_2|| & 0 \\ 0 & a_{11} & 0 \\ 0 & \overline{\alpha}||A_0|| & ||A_2|| \end{pmatrix}.$$

Here α is a complex parameter, a_{11} is real and the relation (6) holds:

$$\left(||A_0||^2 + ||A_2||^2\right)\ \left(1 + |\alpha|^2\right) + |a_{11}|^2 = 1. \tag{6}$$

Both $||A_0||$ and $||A_2||$ are real parameters and α is a complex parameter. (Using a source unitary we can make α real if we wish.)

- *When $d = 3$ the normal form is*

$$\begin{pmatrix} ||A_0|| & a_{10} & -\alpha||A_3|| & 0 \\ 0 & a_{11} & a_{21} & 0 \\ 0 & 0 & a_{22} & 0 \\ 0 & \overline{\alpha}||A_0|| & a_{23} & ||A_3|| \end{pmatrix}.$$

Here α, a_{10}, and a_{23} are complex parameters, $||A_0||$, $||A_3||$, and a_{11} are real parameters, a_{22} is determined and is real, and two equations hold:

$$||A_0||\overline{a_{10}} - \overline{\alpha}a_{10}||A_3|| + a_{11}\overline{a_{21}} + \overline{\alpha}||A_0||\overline{a_{23}} + a_{23}||A_3|| = 0 \tag{7.1}$$

$$\text{trace}(\mathcal{A}^*\mathcal{A}) = \sum |a_{jk}|^2 = 1. \tag{7.2}$$

If $a_{11} \neq 0$, then (7.1) determines a_{21} and (7.2) determines a_{22} (given that it is real and non-negative). Note that the sum in (7.2) includes all the terms (such as $a_{00} = ||A_0||$ and $a_{20} = -\alpha||A_3||$, and so on). If $a_{11} = 0$, then we set $a_{21} = 0$ and again a_{22} is determined.

Proof. The cases where $d = 0$ and $d = 1$ are immediate. When $d = 2$, we have A_0 and A_2 are orthogonal and $A_2 \neq 0$. We may therefore choose the first and third columns as claimed. By (4) and (3) we also must have

$$\langle A_0, A_1\rangle + \langle A_1, A_2\rangle = 0 \tag{8.1}$$

$$||A_0||^2 + ||A_1||^2 + ||A_2||^2 = 1 \tag{8.2}$$

Equation (8.1) becomes

$$||A_0||\overline{a_{10}} + a_{12}||A_3|| = 0$$

which we solve in terms of α. Then we satisfy (8.2) by choosing $|a_{11}|$ appropriately. Using a diagonal unitary map with diagonal elements $1, e^{i\phi}, 1$ for appropriate ϕ we can make a_{11} real and non-negative.

The case when $d = 3$ is similar to the case when $d = 2$ except that we must regard a_{10} and a_{23} as parameters. We then use equations (4) to obtain (7.1) and (7.2). Using a diagonal unitary map we can make the parameter a_{11} real and, after determining $|a_{22}|$, make a_{22} real as well. The normal form follows. Note that there are six parameters, of which $||A_0||, ||A_3||, a_{11}$ are real and α, a_{10}, a_{23} are complex. □

Before proving Theorem 4.1, we discuss **upper-trace identities**.

Definition 4.2. Consider a matrix $\mathcal{V}$ whose column vectors are $V_0, ...V_d$. We say that $\mathcal{V}$ satisfies *upper-trace identities* with parameter values λ_j if

$$\langle V_0, V_d \rangle = \lambda_d.$$

$$\langle V_0, V_{d-1} \rangle + \langle V_1, V_d \rangle = \lambda_{d-1}$$

$$\sum_j \langle V_j, V_{j+k} \rangle = \lambda_k$$

$$\sum_j ||V_j||^2 = \lambda_0.$$

When $\lambda_0 = 1$ and $\lambda_j = 0$ otherwise, these identities are equations (3) and (4) from Proposition 4.1. Their general version facilitates the proof of Theorem 4.1 as follows:

Proof. (of Theorem 4.1) The upper-trace identities are linear equations in the inner products, but quadratic in the entries of the vectors after a basis has been chosen. By regarding some of the entries as parameters we obtain linear equations in the remaining entries. Let f be a polynomial of degree d mapping the unit circle to some unit sphere. Its coefficient vectors A_j and an orthonormal basis determine a matrix $\mathcal{A}$ with entries a_{jk} for $0 \leq j, k \leq d$. Equation (3) is equivalent to $\operatorname{tr}(\mathcal{A}^*\mathcal{A}) = 1$. When $d = 2$ this equation is the same as (6) and when $d = 3$ it is the same as (7.2). We first find a unitary change of coordinates such that the first column of $\mathcal{A}$ is $||A_0||e_0$ and the last column is $||A_d||e_d$ for orthonormal basis elements e_j. Once we have done so, we can only choose unitary maps that preserve these vectors.

Consider the identity $\langle A_0, A_{d-1} \rangle + \langle A_1, A_d \rangle = 0$. There is thus a complex number α such that $a_{1d} = \overline{\alpha}||A_0||$ and $a_{(d-1)0} = -\alpha||A_d||$. The rest of the entries in the first and last row are complex parameters. These parameters are of course subject to the inequalities forced upon us by equation (3) from Proposition 4.1. The entries a_{jk} for $1 \leq j, k \leq d-1$ define a submatrix of size $(d-1)$ by $(d-1)$, called the inner matrix. Let us call the column vectors of this matrix β_j. For $k \geq 0$, the equations (4) now take the form

$$\sum_k \langle \beta_k, \beta_{k+l} \rangle = \lambda_l,$$

where the complex numbers λ_l are expressed in terms of the parameters used in the first and last column and the first and last row of $\mathcal{A}$. Thus the inner $(d-1)$ by $(d-1)$ matrix satisfies upper-trace identities with known parameter values λ_j.

Given a matrix whose columns are the vectors β_j, we choose an orthonormal basis putting it in upper triangular form, and furthermore making the diagonal entries real and non-negative. Doing so depends only upon the inner products, and hence the upper-trace identities are preserved. We have established an induction from degree $d-1$ with no constant term to degree d possibly with a constant term. Hence Theorem 4.1 follows from induction, using as basis steps the cases where the degrees are 1 and 2. □

We next derive some corollaries, including information in the rational case. Put

$$G_{\alpha,r} = (\cos\alpha \frac{r-z}{1-rz}, \sin\alpha, 0, ..., 0)$$

for $\alpha \in [0, \frac{\pi}{2})$ and $r \in [0, 1)$.

Proposition 4.3. *For $N \geq 2$, let $F : \Delta \to \mathbb{B}_N$ be a holomorphic proper rational map of degree one. Then*

1. *F is unitarily equivalent to some $G_{\alpha,r}$.*
2. *If F is polynomial, then it is unitarily equivalent to some $G_{\alpha,0}$.*
3. *For $\alpha, \beta \in [0, \frac{\pi}{2})$ and $0 \leq r_1, r_2 < 1$, the maps G_{α,r_1} and G_{β,r_2} are unitarily equivalent if and only if $\alpha = \beta$ and $r_1 = r_2$.*
4. *A holomorphic rational proper map F of degree one from Δ to $\mathbb{B}_N$ is spherically equivalent to $z \oplus 0$.*

Proof. Part (2) follows directly from Proposition 4.2. Part (3) is a simple computation. We will therefore prove only parts (1) and (4).

To prove part (1), start with $F(z) = \frac{zA+B}{1-\bar{c}z}$, where A, B are vectors and $|c| < 1$. After replacing z by $e^{i\theta}z$, a unitary change in the source, we may assume that $c = r$ is real. We then rewrite the numerator for new vectors A' and B' as

$$zA + B = (z - r)A' + (1 - rz)\ B'.$$

Doing so is possible because the matrix $\begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix}$ is invertible. Then

$$F(z) = \frac{z - r}{1 - rz}A' + B'.$$

Note that A' is multiplied by a sphere map. Since F is a sphere map, $||F(z)||^2 = 1$ on the circle. It follows that $||A'||^2 + ||B'||^2 = 1$ and $\langle A', B' \rangle = 0$. Now we make a unitary change of coordinates in the target such that $A' = (\cos(\alpha), 0, \ldots, 0)$ and $B' = (0, \sin\alpha, 0, \ldots, 0)$. Thus F is unitarily equivalent to $G_{\alpha,r}$.

Finally we prove part (4). By composing F with an automorphism of $\mathbb{B}_N$ if necessary, we may assume $F(0) = 0$ and hence $F(z) = \frac{Az}{(1-bz)}$ for some constant vector A. By applying a unitary map, we may further assume

$$F(z) = \frac{az}{1 - bz} \oplus 0.$$

Since F is a proper map from Δ to $\mathbb{B}_N$, it follows that $\frac{az}{1-bz}$ is a proper map from Δ to Δ. This fact forces $b = 0$ and $a = e^{i\theta}$, and concludes the proof. $\square$

We define a two-parameter family of maps by

$$J_{\alpha,\beta} = C_2 z^2 + C_1 z + C_0$$

where

$$C_2 = (\cos(\alpha)\cos(\beta), 0)$$
$$C_1 = (\sin(\alpha)\sin(\beta), -\sin(\alpha)\cos(\beta))$$
$$C_0 = (0, \cos(\alpha)\sin(\beta)).$$

Here $\alpha, \beta \in [0, \frac{\pi}{2})$. Note that $\langle C_2, C_0 \rangle = 0$, as required by (4).

We define a three-parameter family of maps by

$$J_{\alpha,\beta,\gamma} = (D_2 z^2 + D_1 z + D_0) \oplus 0$$

where

$$D_2 = (\cos(\alpha)\cos(\beta), 0, 0)$$
$$D_1 = (\sin(\alpha)\sin(\beta)\sin(\gamma), \sin(\alpha)\cos(\gamma), -\sin(\alpha)\cos(\beta)\sin(\gamma))$$

$$D_0 = (0, 0, \cos(\alpha)\sin(\beta))\,.$$

Here $\alpha, \beta \in [0, \frac{\pi}{2})$ and $\gamma \in [0, \frac{\pi}{2}]$. Here $\langle D_2, D_0\rangle = 0$, as required by (4).

The following result follows from our normal forms when the degree is 2.

Proposition 4.4. *Let $F : \Delta \to \mathbb{B}_N$ be a holomorphic polynomial proper map of degree two.*

1. *Assume $N = 2$. Then F is unitarily equivalent to some $J_{\alpha,\beta}$.*
2. *For $\alpha_1, \alpha_2, \beta_1, \beta_2$ in $[0, \frac{\pi}{2})$, the maps J_{α_1,β_1} and J_{α_2,β_2} are unitarily equivalent if and only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.*
3. *Assume $N \geq 3$. Then F is unitarily equivalent to some $J_{\alpha,\beta,\gamma}$.*
4. *For $\alpha_1, \alpha_2, \beta_1, \beta_2, \in (0, \frac{\pi}{2}), \gamma_1, \gamma_2 \in [0, \frac{\pi}{2})$, the maps $J_{\alpha_1,\beta_1,\gamma_1}$ and $J_{\alpha_2,\beta_2,\gamma_2}$ are unitarily equivalent if and only if $(\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2)$.*

Remark 4.2. The maps $J_{0,\beta,\gamma}, J_{\alpha,0,\gamma}, J_{\alpha,\beta,\frac{\pi}{2}}$ in part (3) of Proposition 4.4 each have embedding dimension 2.

We next illustrate the proof of Theorem 4.1 in the degree 4 case. Write

$$f(z) = A_0 + A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4.$$

The condition $\langle A_0, A_4\rangle = 0$ enables us to choose coordinates such that

$$\mathcal{A} = \begin{pmatrix} ||A_0|| & * & * & * & 0 \\ 0 & * & * & * & 0 \\ 0 & * & * & * & 0 \\ 0 & * & * & * & 0 \\ 0 & * & * & * & ||A_4|| \end{pmatrix}$$

Now use the condition $\langle A_0, A_3\rangle + \langle A_1, A_4\rangle = 0$ to get

$$\mathcal{A} = \begin{pmatrix} ||A_0|| & ** & ** & -\alpha||A_4|| & 0 \\ 0 & * & * & * & 0 \\ 0 & * & * & * & 0 \\ 0 & * & * & * & 0 \\ 0 & \overline{\alpha}||A_0|| & ** & ** & ||A_4|| \end{pmatrix}$$

Regard the ** as complex parameters. The rest of the identities become upper-trace identities on the inner three-by-three matrix. Since everything depends only on the inner products, things are unitarily invariant. We can then use a unitary to put the inner matrix in upper-triangular form:

$$\begin{pmatrix} ||A_0|| & ** & ** & -\alpha||A_4|| & 0 \\ 0 & * & * & * & 0 \\ 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & \overline{\alpha}||A_0|| & ** & ** & ||A_4|| \end{pmatrix}.$$

The inner matrix may be regarded as 3 column vectors $\beta_1, \beta_2, \beta_3$. The upper-trace identities express

$$||\beta_1||^2 + ||\beta_2||^2 + ||\beta_3||^2 \tag{9.1}$$

$$\langle \beta_1, \beta_2\rangle + \langle \beta_2, \beta_3\rangle \tag{9.2}$$

$$\langle \beta_1, \beta_3\rangle \tag{9.3}$$

as known quantities in terms of the various parameters along the first and last row of $\mathcal{A}$. There are 5 complex and 2 real parameters along the first and last row.

There are 6 unknown parameters in the inner matrix, but these satisfy 3 equations arising from expressing (9.1), (9.2), (9.3) in terms of the known parameters. Hence there remain 3 new parameters. By using diagonal unitaries we obtain in total 10 parameters of which 4 are real.

5. HOMOTOPY

Let $\mathcal{H}(n, N)$ denote the set of homotopy equivalence classes of rational proper holomorphic maps from $\mathbb{B}_n$ to $\mathbb{B}_N$. We write $h(n, N)$ for the cardinality of $\mathcal{H}(n, N)$. We recall some basic facts about these homotopy classes. Each class in $\mathcal{H}(1, 1)$ is given by z^k for a positive integer k. When $N \geq n \geq 2$, the cardinality $h(n, N)$ is always finite. See [DL1]. Of course, when $N < n$, the set is empty. Very few of the the $h(n, N)$ are known. We have $h(2, 3) = 4$. When $n \geq 3$, we have $h(n, N) = 1$ if $n \leq N \leq 2n - 2$ and $h(n, 2n - 1) = 2$. If $n \geq 4$, then $h(n, 2n) = 1$. We next establish the following new result along these lines.

Theorem 5.1. *For $N \geq 2$, all holomorphic rational proper maps from Δ to $\mathbb{B}_N$ belong to one homotopy equivalence class. Thus $h(1, N) = 1$ if $N \geq 2$.*

Lemma 5.1. *Let F and G be homotopic proper maps from Δ to some $\mathbb{B}_N$. Let ζ be a proper map from Δ to itself. Then ζF and ζG are also homotopic proper maps from Δ to $\mathbb{B}_N$. Similarly, if $(f_1, ..., f_N)$ and $(g_1, ..., g_N)$ are homotopic via polynomial sphere maps from Δ to $\mathbb{B}_N$, and ζ is a polynomial proper map, then $(f_1, ..., f_{N-1}, \zeta f_N)$ and $(g_1, ..., g_{N_1}, \zeta g_N)$ are homotopic via polynomial proper maps.*

Proof. Suppose for $t \in [0, 1]$ that H_t is a family of proper maps, depending continuously on t, with $H_0 = F$ and $H_1 = G$. Then ζH_t also depends continuously on t. Since ζ maps the circle to itself, $||\zeta H_t||^2 = ||H_t||^2$ on the circle, and hence ζH_t is also proper. The second statement follows from a similar argument, assuming $\zeta = z^m$. □

Lemma 5.2. *Let $F = \frac{p}{q} = \frac{(p_1,...,p_N)}{q}$ be a holomorphic rational proper map from Δ to $\mathbb{B}_N$ that is reduced to lowest terms with $F(0) = 0$. Then* $\deg(p) > \deg(q)$.

Proof. Assume F has degree d. Put $p(z) = \sum_{j=0}^{d} P_j z^j$ and $q(z) = \sum_{j=0}^{d} q_j z^j$. The condition $||p(z)||^2 = |q(z)|^2$ on the sphere yields upper-trace identities as in Proposition 4.1. Assuming that $q(0) = 1$, one of these identities gives $\langle P_d, P_0 \rangle = q_d$. Since P_0 is assumed to be 0, the conclusion follows. □

Proof. (of Theorem 5.1) Let F be any holomorphic rational proper map from Δ to $\mathbb{B}_N$. It suffices to show that F is homotopic to $z \oplus 0$. Write $d = \deg(F)$. We will prove the result by induction on the degree. Part 4 of Proposition 4.3 shows that the statement holds for $d = 1$. We assume it holds for $d \leq k$ for some $k \geq 1$. Now consider F of degree $d = k+1$. Note that F is spherically equivalent and thus homotopic to a rational proper map F_1 with $F_1(0) = 0$ and $\deg(F_1) = d = k + 1$. Put $F_1 = zF_2$. It follows from Lemma 5.2 that $\deg(F_2) = k$. By the induction hypothesis, F_2 is homotopic to $z \oplus 0$. We conclude by Lemma 5.1 that $F_1 = zF_2$ is homotopic to $z^2 \oplus 0$ which is also homotopic to $z \oplus 0$. Theorem 5.1 follows. □

We write $\mathcal{S}(N, d)$ for the solution set of the system from Proposition 4.1:

$$\mathcal{S}(N, d) = \{(A_0, ..., A_d) : \text{ equations (3) and (4) are satisfied}\}.$$

Let $\mathcal{P}(N,d)$ be the collection of all holomorphic polynomial sphere maps from Δ to $\mathbb{C}^N$ of degree at most d. There is a one-to-one correspondence between $\mathcal{S}(N,d)$ and $\mathcal{P}(N,d)$.

The following result shows that $\mathcal{S}(N,d)$ is a connected subvariety of $\mathbb{C}^{N(d+1)}$ when $N \geq 2$.

Proposition 5.1. *Assume $N \geq 2$. Any two polynomial sphere maps from Δ to $\mathbb{C}^N$ of degree d are homotopic via polynomial sphere maps.*

Proof. The conclusion follows if $\mathcal{P}(N,d)$ is path-connected. We prove the path-connectedness in the next lemma.

Lemma 5.3. *$\mathcal{P}(N,d)$ is path-connected for any $N \geq 2$ and $d \geq 0$.*

Proof. We prove the lemma by induction on d. Proposition 4.2 shows that the lemma holds for $d=0$ and $d=1$. We assume for some $k \geq 1$ that it is true for $d \leq k$ for and we will prove it for $d=k+1$. Fix $f \in \mathcal{P}(N,d)$ with $d=k+1$. Write $f=\sum_{j=0}^{d} A_j z^j$ as usual; here we regard the A_j as column vectors. Note that f has N components. It suffices to show f is homotopic to $z \oplus 0$ in $\mathcal{P}(N,d)$. Write

$$A=(A_0,\ldots,A_d)\,.$$

We may assume f is of degree d and hence $A_d \neq 0$. Find a unitary matrix U such that $UA_d=(0,...,0,||A_d||)$. Put $B=UA=(B_0,...,B_d)$ and set $g=Uf$. Since the unitary group is path-connected, g is homotopic to f. Moreover, by the form of B_d, each g_j is of degree at most $d-1=k$ except for g_N which is of degree d. We still have $\langle B_0, B_d\rangle = 0$. Hence the last component of B_0 is zero and thus g_N has no constant term. Write $g_N = zu_N$. We set

$$h=(g_1,...,g_{N-1},u_N).$$

Note that $h \in \mathcal{P}(N,d-1)$. It follows from the induction hypothesis that h is homotopic to $z \oplus 0$ in $\mathcal{P}(N,d-1)$. It follows from Lemma 5.1 that g (and thus f) is homotopic to $z^2 \oplus 0$ and thus to $z \oplus 0$ in $\mathcal{P}(N,d)$. Lemma 5.3 follows. As noted above, the proposition follows from this lemma. □

We naturally ask the following question in the general setting.

Question: *Assume $f,g : \mathbb{B}_n \to \mathbb{B}_N$ are homotopic polynomial proper maps. Can we always find a homotopy where each intermediate map is also a polynomial?*

The answer is affirmative in some special cases. If $3 \leq n \leq N \leq 2n$, for example, then all the polynomial maps are known and the result holds. When $n=2$ and $N=4$, there are many more polynomial maps and the precise homotopy relations are not known.

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