The β constant appeared in algebraic and complex geometry

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Nevanlinna theory: Introduction the notations

Let X be a complex projective variety and let D be an effective Cartier divisor. Let s_D be the canonical section of $[D]$ (i.e. $[s_D = 0] = D$) and $\| \ \|$ be an hemitian metric, i.e. $\|s\|^2 = |s_\alpha|^2 h_\alpha$.

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$$m_f(r, D) + N_f(r, D) = T_{f,D}(r) + O(1)$$

where $\lambda_D(x) = -\log \|s_D(x)\| = -$ log distance from x to D (Weil function for D), $m_f(r, D) = \int_0^{2\pi} \lambda_D(f(re^{i\theta}))\frac{d\theta}{2\pi}$ (Approximation function). $T_{f,L}(r) := \int_1^r \frac{dt}{t} \int_{|z|<t} f^*c_1(L)$ (Height function).

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Nevanlinna's SMT for meromorphic functions

The Second Main Theorem(Nevanlinna, 1929). Let f be meromorphic (non-constant) on $\mathbb{C}$ and $a_1, ..., a_q \in \mathbb{C} \cup \{\infty\}$ distinct. Then, for any $\epsilon > 0$, $(q - 2 - \epsilon)T_f(r) \leq_{exc} \sum_{j=1}^{q} N_f(r, a_j)$, or equivalently

$$\sum_{j=1}^{q} m_f(r, a_j) \leq_{exc} (2 + \epsilon)T_f(r) \quad ,$$

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Cartan's Theorem (1933). Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic map. Let $H_1, \ldots, H_q$ be the hyperplanes in general position on $\mathbb{P}^n(\mathbb{C})$. Then, for any $\epsilon > 0$, $\sum_{j=1}^{q} m_f(r, H_j) \leq_{exc} (n + 1 + \epsilon) T_f(r)$.

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In 2004, Ru extended the above result to hypersurfaces for $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ with Zariski dense image.
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Theorem (Ru, 2009). Let $f : \mathbb{C} \to X$ be holo and Zariski dense, $D_1, \ldots, D_q$ be divisors in general position in X. Assume that $D_j \sim d_j A$ (A being ample). Then, for $\forall\ \epsilon > 0$,

$$\sum_{j=1}^q \frac{1}{d_j} m_f(r, D_j) \leq_{exc} (\dim X + 1 + \epsilon) T_{f,A}(r)$$

Theorem (Ru-Vojta, Amer. J. Math., 2020). Let X be a smooth complex projective variety and let $D_1, \ldots, D_q$ be effective Cartier divisors in general position. Let $D = D_1 + \cdots + D_q$. Let $\mathscr{L}$ be a line sheaf on X with $h^0(\mathscr{L}^N) \geq 1$ for N big enough. Let $f : \mathbb{C} \to X$ be a holomorphic map with Zariski image. Then, for every $\epsilon > 0$,

$$\sum_{j=1}^{q} \beta_j(\mathscr{L}, D_j) m_f(r, D_j) \leq_{exc} (1+\epsilon) T_{f,\mathscr{L}}(r)$$

where

$$\beta(\mathscr{L}, D) = \limsup_{N \to +\infty} \frac{\sum_{m \geq 1} \dim H^0(X, \mathscr{L}^N(-mD))}{N \dim H^0(X, \mathscr{L}^N)}.$$

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In the case when $D_j \sim A$, then $\beta(D, D_j) = \frac{q}{n+1}$, where $D = D_1 + \cdots + D_q$.

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The Basic Theorem. Let X be a complex projective variety and let $\mathcal{L}$ be a line sheaf on X with $\dim H^0(X, \mathcal{L}) \geq 1$. Let $s_1, \ldots, s_q \in H^0(X, \mathcal{L})$. Let $f : \mathbf{C} \to X$ be a holomorphic map with Zariski-dense image. Then, for any $\epsilon > 0$,

$$\int_0^{2\pi} \max_J \sum_{j \in J} \lambda_{s_j}(f(re^{i\theta})) \frac{d\theta}{2\pi} \leq_{exc} (\dim H^0(X, \mathcal{L}) + \epsilon) T_{f,\mathcal{L}}(r)$$

where the set J ranges over all subsets of $\{1, \ldots, q\}$ such that the sections $(s_j)_{j \in J}$ are linearly independent.

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where the set J ranges over all subsets of $\{1, \ldots, q\}$ such that the sections $(s_j)_{j \in J}$ are linearly independent. Note: The $D \sim_{\mathbb{Q}} L$ is of m-basis type if $D := \frac{1}{mN_m} \sum_{s \in \mathcal{B}} (s)$, where $\mathcal{B}$ is a basis of $H^0(X, \mathcal{L}^{\otimes m})$, where $N_m = \dim H^0(X, \mathcal{L}^{\otimes m})$.

Theorem (Weak version of Ru-Vojta). Let X be a complex projective variety and let $D_1, \ldots, D_q$ be effective Cartier divisors such that at most ℓ of such divisors meet at any point of X. Let $\mathcal{L}$ be a line sheaf on X with $h^0(\mathcal{L}^N) \geq 1$ for N big enough. Let $f : \mathbf{C} \to X$ be a holomorphic map with Zariski-dense image. Then, for every $\epsilon > 0$, $\sum_{j=1}^q \beta(\mathcal{L}, D_j) m_f(r, D_j) \leq_{exc} \ell\,(1+\epsilon)\, T_{f,\mathcal{L}}(r)$.

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The proof is using the Basic Theorem by choosing a a suitable m-basis of $H^0(X, \mathcal{L}^m)$ through a filtration.

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- Consider the following filtration of $H^0(X, \mathcal{L}^N)$:

$$H^0(X, \mathcal{L}^N) \supseteq H^0(X, \mathcal{L}^N(-D_{i_0})) \supseteq \cdots \supseteq H^0(X, \mathcal{L}^N(-mD_{i_0})) \supseteq \cdots$$

and choose a basis $s_1, \cdots, s_l \in H^0(X, \mathcal{L}^N)$, where $l = h^0(\mathcal{L}^N)$ according to this filtration.

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$$\begin{aligned} \sum_{j=1}^{l}(s_j) &\geq \left(\sum_{m=0}^{\infty} m[h^0(\mathcal{L}^N(-mD_{i_0})) - h^0(\mathcal{L}^N(-(m+1)D_{i_0}))]\right) D_{i_0} \\ &= \left(\sum_{m=1}^{\infty} h^0(\mathcal{L}^N(-mD_{i_0}))\right) D_{i_0}. \end{aligned}$$

Hence the *m*-basis

$$\frac{1}{Nh^0(\mathcal{L}^N)} \sum_{j=1}^{h^0(\mathcal{L}^N)} (s_j)$$

$$\geq \frac{\sum_{m=1}^{\infty} h^0(\mathcal{L}^N(-mD_{i_0})}{Nh^0(\mathcal{L}^N)} D_{i_0}.$$

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It then follows from the Basic Theorem. In summary: The proof is about estimate the order of the m-basis coming from the filtration, and then apply the basic Theroem.

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Theorem (Roth, 1955). Let α be an algebraic number of degree≥ 2. Then, for any given $\varepsilon > 0$, we have $\left|\alpha - \frac{p}{q}\right| > \frac{1}{q^{2+\epsilon}}$ for all, but finitely many, coprime integers p and q.

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Roth's Theorem. k=number field and S=finite set of places on k. $a_1, \dots, a_q$ distinct in $\mathbb{P}^1(k)$. Then

$$\sum_{j=1}^{q} \sum_{v \in S} \log^+ \frac{1}{\|x - a_j\|_v} \leq (2+\epsilon) h(x)$$

holds for $\forall\ x \in \mathbb{P}^1(k)$ except for finitely many points.

Denote by

$$m_S(x, a) := \sum_{v \in S} \log^+ \frac{1}{\|x - a\|_v}.$$

Then $\sum_{j=1}^{q} m_S(x, a_j) \leq_{exc} (2+\epsilon) h(x)$.

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$$\beta(L, D) := \limsup_{N\to\infty} \frac{\sum_{m\geq 1} h^0(L^N(-mD))}{Nh^0(L^N)}.$$

Theorem (Ru-Vojta, 2020) [Arithmetic Part] Let X be a projective variety over a number field k, and $D_1, \ldots, D_q$ be effective Cartier divisors intersecting properly on X. Let $S \subset M_k$ be a finite set of places. Then, for every $\epsilon > 0$, the inequality

$$\sum_{i=1}^{q} \beta(L, D_j) m_S(x, D_j) \leq (1+\epsilon) h_L(x)$$

holds for all k-rational points outside a proper Zariski-closed subset of X.

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$$Vol(L) = \limsup_{m \to \infty} \frac{\dim H^0(X, mL)}{m^n/n!}$$

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Notice that $Vol(kL) = k^n Vol(L)$ so the volume function can be extended to $\mathbb{Q}$-divisors. Also note that $Vol(\ )$ depends only on the numerical class of L, so it is defined on $NS(X) := Div(X)/Num(X)$ and extends uniquely to a continuous function on $NS(X)_{\mathbb{R}}$. The volume function lies at the intersection of many fields of mathematics and has a variety of interesting applications (bi-rational geometry, complex geometry, number theory etc.)

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So we can express the above constant through the notion of Vol(L),

$$\beta(L,D) = \frac{1}{Vol(L)} \int_0^\infty Vol(L-tD)dt.$$

This can be proved by using the theory of Okounkov body.

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Let L be a big line bindule on X. An Okounkov body $\Delta(L) \subset \mathbb{R}^n$ (where $n = \dim X$) is a compact convex set designed to study the asymptotic behavior of $H^0(X, mL)$, as $m \to \infty$. They have the crucial property that the Eulidean volume $\text{Vol}(\Delta) = \lim_{m\to\infty} \frac{\dim H^0(X,mL)}{m^n} = \frac{\text{Vol(L)}}{n!}$.

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$$\Delta = \Sigma \cap (\{1\} \times \mathbb{R}^n) \subset \mathbb{R}^n.$$

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The Vanishing sum: Given a filtration $\mathcal{F}$ (for example $\mathcal{F}_m^t := H^0(mL - tD)$), consider the *jumping numbers* $0 \leq a_{m,1} \leq \cdots \leq a_{m,N_m}$, defined by,
$a_{m,j} = a_{m,j}^{\mathcal{F}} = \inf\{t \in \mathbb{R}_+ \mid \mathrm{codim}\mathcal{F}_m^t \geq j\}$ for $1 \leq j \leq N_m$.

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$$\lim_{m\to+\infty} \mu_m = \mu$$

in the weak sense of measures on $\mathbb{R}_+$, where $\mu = (G_{\mathcal{F}})_*\lambda$, $G_{\mathcal{F}} : \Delta(V_\bullet) \to [-\infty, +\infty)$, $G_{\mathcal{F}}(x) := \sup\{t \in \mathbb{R}, x \in \Delta(V_\bullet^t)\}$.

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Blum-Jonsson used m-basis type to describe the stability threshold $\delta(L)$: they proved $\delta(L) = \lim \delta_m(L)$, where $\delta_m(L) := \inf\{\text{lct}(D) \mid D \sim_{\mathbb{Q}} L \text{ of m-basis type}\}$. (through m-basis).
Algebraic geometry definition of "log canonical threshold":

$$\text{lct}(D) = \min_E \frac{A_X(E)}{\text{ord}_E(D)},$$

where the minimal is taken over all primes E over X.

The log canonical threshold through singular metric

Tian in 1987 introduced $\alpha(L)$ the log canonical threshold of L as follows: Let $h = e^{-\phi}$ be a singular metric with $\Theta_{L,h} \geq 0$, where $\Theta_{L,h} = \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log\phi$. Define $c_p(h) = \sup\{c \mid e^{-2c\phi}$ is locally integrable at p $\}$. Define, for $p \in X$, $\alpha_p(L) = \inf_{h:\Theta_{L,h}\geq 0} c_p(h)$ and $\alpha(L) = \inf_{p\in X} \alpha_p(L)$.

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Proof of Blum-Jonsson's result

To see Blum-Jonsson's result: $\lim_{m\to\infty}\delta_m(L)=\delta(L)$, where $\delta(L)=\inf_E \frac{A_X(E)}{\beta(L,E)}$, $\delta_m(L):=\inf\{\mathrm{lct}(D)\mid D\sim_{\mathbb{Q}} L \text{ of m-basis type}\}$, $\mathrm{lct}(D)=\min_E \frac{A_X(E)}{\mathrm{ord}_E(D)}$,

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The choice of m-basis

Let E be an effective Cartier divisor. The m-basis comes from the filtration $\mathcal{F}_m^t = H^0(X, mL - tE), t \geq 0$ of $H^0(X, mL)$. The m-basis is $D := \frac{1}{mN_m}\sum_{s\in B}(s)$. Notice that, for any $s \in W_t := H^0(X, mL - tE)$, $\mathrm{ord}_E(s) \geq t$, so $\mathrm{ord}_E(D) =$

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$$= \frac{1}{mN_m}\left(\sum_{t=1}^{\infty}\dim W_t\right) \to \beta(L, E) \text{ as } m \to \infty.$$

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Theorem.

$$\delta(L) \leq \frac{1}{\max_{1 \leq i \leq q} \beta(D_i, L)} \text{lct}(D),$$

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- Furthermore, $\alpha(L) = \inf_E \frac{A(E)}{T(L,E)}$. This gives (B) (as above)

$$\alpha(L) \leq \delta(L) \leq (n+1)\alpha(L).$$