

# Hermitian-Yang-Mills approach to the conjecture of Griffiths on the positivity of ample vector bundles

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## Ample vector bundles

Let  $X$  be a projective  $n$ -dimensional manifold and  $E \rightarrow X$  a holomorphic vector bundle of rank  $r \geq 1$ .

### Ample vector bundles

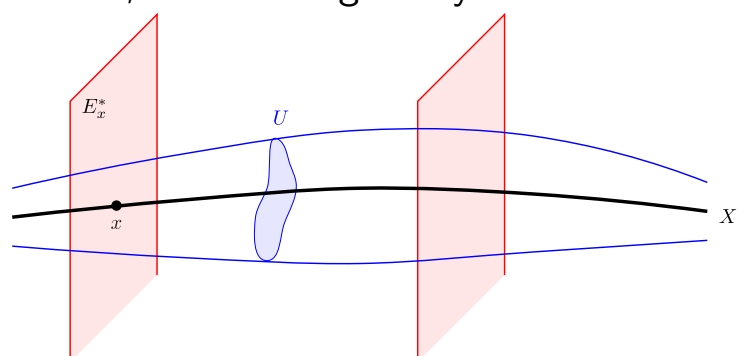
$E \rightarrow X$  is said to be **ample in the sense of Hartshorne** if the associated line bundle  $\mathcal{O}_{\mathbb{P}(E)}(1)$  on  $\mathbb{P}(E)$  is ample, i.e. by Kodaira  $\iff \exists \mathcal{C}^\infty$  **hermitian metric on  $\mathcal{O}_{\mathbb{P}(E)}(1)$  with positive curvature.**

This is equivalent to the existence of a strongly pseudoconvex tubular neighborhood  $U$  of the 0-section in  $E^*$ , i.e. of a negatively curved Finsler metric on  $E^*$ .

Geometric interpretation:

$U$  can be taken  $S^1$  invariant

$$U \rightsquigarrow \bigcap_{|\lambda|=1} \lambda U$$



# Chern curvature tensor and positivity concepts

## Chern curvature tensor

This is  $\Theta_{E,h} = i\nabla_{E,h}^2 \in C^\infty(\Lambda^{1,1} T_X^* \otimes \text{Hom}(E, E))$ , written

$$\Theta_{E,h} = i \sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu$$

in terms of an orthonormal frame  $(e_\lambda)_{1 \leq \lambda \leq r}$  of  $E$ .

## Griffiths and Nakano positivity

One looks at the associated quadratic form on  $S = T_X \otimes E$

$$\tilde{\Theta}_{E,h}(\xi \otimes v) := \langle \Theta_{E,h}(\xi, \bar{\xi}) \cdot v, v \rangle_h = \sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{jk\lambda\mu} \xi_j \bar{\xi}_k v_\lambda \bar{v}_\mu.$$

Then  $E$  is said to be:

- Griffiths positive (Griffiths 1969) if at any point  $z \in X$

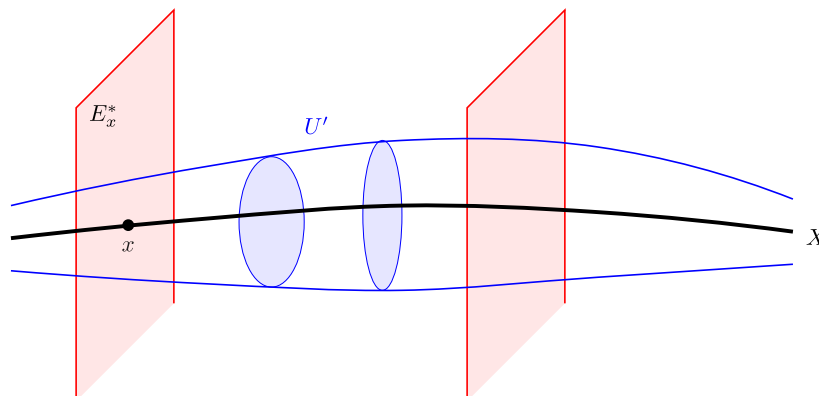
$$\tilde{\Theta}_{E,h}(\xi \otimes v) > 0, \quad \forall 0 \neq \xi \in T_{X,z}, \quad \forall 0 \neq v \in E_z$$

- Nakano positive (Nakano 1955) if at any point  $z \in X$

$$\tilde{\Theta}_{E,h}(\tau) = \sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{jk\lambda\mu} \tau_{j,\lambda} \bar{\tau}_{k,\mu} > 0, \quad \forall 0 \neq \tau \in T_{X,z} \otimes E_z.$$

## Geometric interpretation of Griffiths positivity

**Griffiths positivity** of  $E$  is equivalent to the existence of a strongly pseudoconvex neighborhood  $U'$  of the 0-section in  $E^*$  whose fibers are (varying) **hermitian balls**.



(**Nakano**  $> 0$  is more restrictive than strict pseudoconvexity of  $U'$ .)

## Easy and well known facts

$$E \text{ Nakano positive} \Rightarrow E \text{ Griffiths positive} \Rightarrow E \text{ ample.}$$

In fact,  $E$  Griffiths positive  $\Rightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$  positive.

# Dual Nakano positivity – a conjecture

## Curvature tensor of the dual bundle $E^*$

$$\Theta_{E^*,h} = -{}^T\Theta_{E,h} = - \sum_{1 \leq j,k \leq n, 1 \leq \lambda, \mu \leq r} c_{jk\mu\lambda} dz_j \wedge d\bar{z}_k \otimes (e_\lambda^*)^* \otimes e_\mu^*.$$

## Dual Nakano positivity

One requires

$$-\tilde{\Theta}_{E^*,h}(\tau) = \sum_{1 \leq j,k \leq n, 1 \leq \lambda, \mu \leq r} c_{jk\mu\lambda} \tau_{j\lambda} \bar{\tau}_{k\mu} > 0, \quad \forall 0 \neq \tau \in T_{X,z} \otimes E_z^*.$$

Dual Nakano positivity is **clearly stronger** than Griffiths positivity.

Also, it is better behaved than Nakano positivity, e.g.

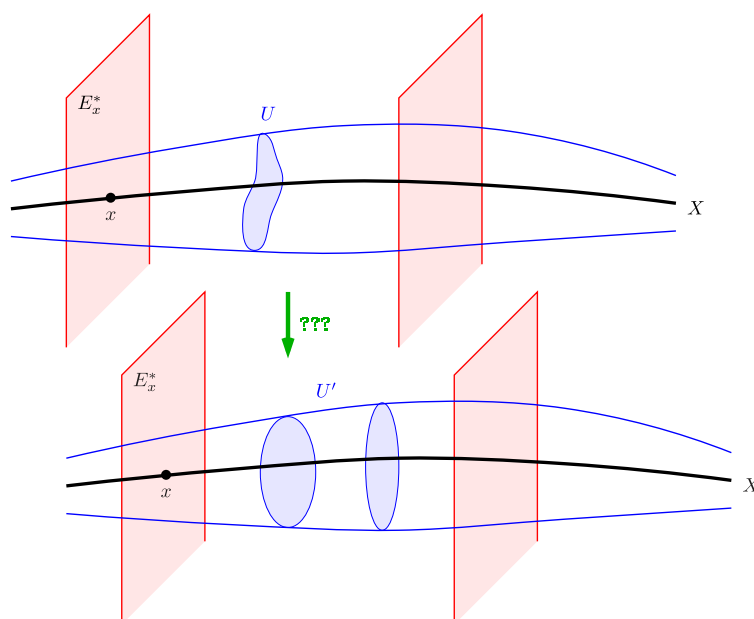
$E$  dual Nakano positive

$\Rightarrow$  any quotient  $Q = E/S$  is also dual Nakano positive.

(Very speculative) conjecture

Is it true that  $E$  ample  $\Rightarrow E$  dual Nakano positive ?

## Geometric interpretation of the conjecture



**Basic question.** Is there a (geometric, analytic) procedure that turns the strictly pseudoconvex neighborhood  $U$  into another strictly pseudoconvex  $U'$  that would be a **ball bundle** ?

**Answer is yes if  $n = \dim X = 1$**  (Umemura, Campana-Flenner) !!

# Brief discussion around this positivity conjecture

If true, Griffiths conjecture would follow:

$$E \text{ ample} \Leftrightarrow E \text{ dual Nakano positive} \Leftrightarrow E \text{ Griffiths positive.}$$

Remark

$$E \text{ ample} \not\Rightarrow E \text{ Nakano positive, in fact} \\ E \text{ Griffiths positive} \not\Rightarrow E \text{ Nakano positive.}$$

For instance,  $T_{\mathbb{P}^n}$  is easily shown to be ample and Griffiths positive for the Fubini-Study metric, but it is **not Nakano positive**. Otherwise the Nakano vanishing theorem would then yield

$$H^{n-1, n-1}(\mathbb{P}^n, \mathbb{C}) = H^{n-1}(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{n-1}) = H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) = 0 \quad !!!$$

Let us mention here that there are already known subtle relations between ampleness, Griffiths and Nakano positivity are known to hold – for instance, B. Berndtsson has proved that the ampleness of  $E$  implies the Nakano positivity of  $S^m E \otimes \det E$  for every  $m \in \mathbb{N}$ .

## “Total” determinant of the curvature tensor

If the Chern curvature tensor  $\Theta_{E,h}$  is **dual Nakano positive**, then one can introduce the  $(n \times r)$ -dimensional determinant of the corresponding Hermitian quadratic form on  $T_X \otimes E^*$

$$\det_{T_X \otimes E^*}({}^T \Theta_{E,h})^{1/r} := \det(c_{jk\mu\lambda})_{(j,\lambda),(k,\mu)}^{1/r} idz_1 \wedge d\bar{z}_1 \wedge \dots \wedge idz_n \wedge d\bar{z}_n.$$

This  $(n, n)$ -form does not depend on the choice of coordinates  $(z_j)$  on  $X$ , nor on the choice of the orthonormal frame  $(e_\lambda)$  on  $E$ .

Basic idea

Assigning a “matrix Monge-Ampère equation”

$$\det_{T_X \otimes E^*}({}^T \Theta_{E,h})^{1/r} = f > 0$$

where  $f$  is a positive  $(n, n)$ -form, may enforce the dual Nakano positivity of  $\Theta_{E,h}$  if that assignment is combined with a continuity technique from an initial starting point where positivity is known.

# Continuity method (case of rank 1)

For  $r = 1$  and  $h = h_0 e^{-\varphi}$ , we have

$${}^T \Theta_{E,h} = \Theta_{E,h} = -i\partial\bar{\partial} \log h = \omega_0 + i\partial\bar{\partial}\varphi,$$

and the equation reduces to a standard Monge-Ampère equation

$$(*) \quad (\Theta_{E,h})^n = (\omega_0 + i\partial\bar{\partial}\varphi)^n = f.$$

If  $f$  is given and independent of  $h$ , Yau's theorem guarantees the existence of a unique solution  $\theta = \Theta_{E,h} > 0$ , provided  $E$  is an ample line bundle and  $\int_X f = c_1(E)^n$ .

When the right hand side  $f = f_t$  of  $(*)$  varies smoothly with respect to some parameter  $t \in [0, 1]$ , one then gets a smoothly varying solution

$$\Theta_{E,h_t} = \omega_0 + i\partial\bar{\partial}\varphi_t > 0,$$

and the positivity of  $\Theta_{E,h_0}$  forces the positivity of  $\Theta_{E,h_t}$  for all  $t$ .

## Undeterminacy of the equation

Assuming  $E$  to be ample of rank  $r > 1$ , the equation

$$(**) \quad \det_{T_X \otimes E^*} ({}^T \Theta_{E,h})^{1/r} = f > 0$$

becomes underdetermined, as the real rank of the space of hermitian matrices  $h = (h_{\lambda\mu})$  on  $E$  is equal to  $r^2$ , while  $(**)$  provides only 1 scalar equation.

(Solutions might still exist, but lack uniqueness and a priori bounds.)

### Conclusion

In order to recover a well determined system of equations, one needs an additional “matrix equation” of rank  $(r^2 - 1)$ .

### Observation 1 (from the Donaldson-Uhlenbeck-Yau theorem)

Take a Hermitian metric  $\eta_0$  on  $\det E$  so that  $\omega_0 := \Theta_{\det E, \eta_0} > 0$ . If  $E$  is  $\omega_0$ -polystable,  $\exists h$  Hermitian metric  $h$  on  $E$  such that

$$\omega_0^{n-1} \wedge \Theta_{E,h} = \frac{1}{r} \omega_0^n \otimes \text{Id}_E \quad (\text{Hermite-Einstein equation, slope } \frac{1}{r}).$$

## Resulting trace free condition

### Observation 2

The trace part of the above Hermite-Einstein equation is “automatic”, hence the equation is equivalent to the trace free condition

$$\omega_0^{n-1} \wedge \Theta_{E,h}^\circ = 0,$$

when decomposing any endomorphism  $u \in \text{Herm}(E, E)$  as

$$u = u^\circ + \frac{1}{r} \text{Tr}(u) \text{Id}_E \in \text{Herm}^\circ(E, E) \oplus \mathbb{R} \text{Id}_E, \quad \text{tr}(u^\circ) = 0.$$

### Observation 3

The trace free condition is a matrix equation of **rank  $(r^2 - 1)$**  !!!

### Remark

In case  $\dim X = n = 1$ , the trace free condition means that  $E$  is **projectively flat**, and the Umemura proof of the Griffiths conjecture proceeds exactly in that way, using the fact that the graded pieces of the Harder-Narasimhan filtration are projectively flat.

## Towards a “cushioned” Hermite-Einstein equation

In general, one cannot expect  $E$  to be  $\omega_0$ -polystable, but Uhlenbeck-Yau have shown that there always exists a smooth solution  $q_\varepsilon$  to a certain **“cushioned” Hermite-Einstein equation**.

To make things more precise, let  $\text{Herm}(E)$  be the space of Hermitian (non necessarily positive) forms on  $E$ . Given a reference Hermitian metric  $H_0 > 0$ , let  $\text{Herm}_{H_0}(E, E)$  be the space of  $H_0$ -Hermitian endomorphisms  $u \in \text{Hom}(E, E)$ ; denote by

$$\text{Herm}(E) \xrightarrow{\cong} \text{Herm}_{H_0}(E, E), \quad q \mapsto \tilde{q} \text{ s.t. } q(v, w) = \langle \tilde{q}(v), w \rangle_{H_0}$$

the natural isomorphism. Let also

$$\text{Herm}_{H_0}^\circ(E, E) = \{q \in \text{Herm}_{H_0}(E, E); \text{tr}(q) = 0\}$$

be the subspace of “trace free” Hermitian endomorphisms.

In the sequel, we fix  $H_0$  on  $E$  such that

$$\Theta_{\det E, \det H_0} = \omega_0 > 0.$$

## Uhlenbeck-Yau 1986, Theorem 3.1

For every  $\varepsilon > 0$ , there **always exists** a (unique) smooth Hermitian metric  $q_\varepsilon$  on  $E$  such that

$$\omega_0^{n-1} \wedge \Theta_{E, q_\varepsilon} = \omega_0^n \otimes \left( \frac{1}{r} \text{Id}_E - \varepsilon \log \tilde{q}_\varepsilon \right),$$

where  $\tilde{q}_\varepsilon$  is computed with respect to  $H_0$ , and  $\log g$  denotes the logarithm of a positive Hermitian endomorphism  $g$ .

The reason is that the term  $-\varepsilon \log \tilde{q}_\varepsilon$  is a “friction term” that prevents the explosion of the a priori estimates, similarly what happens for Monge-Ampère equations  $(\omega_0 + i\partial\bar{\partial}\varphi)^n = e^{\varepsilon\varphi+f}\omega_0^n$ .

The above matrix equation is equivalent to prescribing  $\det q_\varepsilon = \det H_0$  and the trace free equation of rank  $(r^2 - 1)$

$$\omega_0^{n-1} \wedge \Theta_{E, q_\varepsilon}^\circ = -\varepsilon \omega_0^n \otimes \log \tilde{q}_\varepsilon.$$

## Search for an appropriate evolution equation

### General setup

In this context, given  $\alpha > 0$  large enough, it is natural to search for a time dependent family of metrics  $h_t(z)$  on the fibers  $E_z$  of  $E$ ,  $t \in [0, 1]$ , satisfying a generalized Monge-Ampère equation

$$(D) \quad \det_{T_X \otimes E^*} \left( {}^T \Theta_{E, h_t} + (1-t)\alpha \omega_0 \otimes \text{Id}_{E^*} \right)^{1/r} = f_t \omega_0^n, \quad f_t > 0,$$

and trace free, rank  $r^2 - 1$ , Hermite-Einstein conditions

$$(T) \quad \omega_t^{n-1} \wedge \Theta_{E, h_t}^\circ = g_t$$

with smoothly varying families of functions  $f_t \in C^\infty(X, \mathbb{R})$ , Hermitian metrics  $\omega_t > 0$  on  $X$  and sections

$$g_t \in C^\infty(X, \Lambda_{\mathbb{R}}^{n,n} T_X^* \otimes \text{Herm}_{h_t}^\circ(E, E)), \quad t \in [0, 1].$$

Observe that this is a determined (not overdetermined!) system.

## Choice of the initial state ( $t = 0$ )

We start with the Uhlenbeck-Yau solution  $h_0 = q_\varepsilon$  of the “cushioned” trace free Hermite-Einstein equation, so that  $\det h_0 = \det H_0$ , and take  $\alpha > 0$  so large that

$${}^T\Theta_{E,h_0} + \alpha\omega_0 \otimes \text{Id}_{E^*} > 0 \text{ in the sense of Nakano.}$$

If conditions (D) and (T) can be met for all  $t \in [0, 1]$ , thus without any explosion of the solutions  $h_t$ , we infer from (D) that

$${}^T\Theta_{E,h_t} + (1-t)\alpha\omega_0 \otimes \text{Id}_{E^*} > 0 \text{ in the sense of Nakano}$$

for all  $t \in [0, 1]$ .

### Observation

At time  $t = 1$ , we would then get a Hermitian metric  $h_1$  on  $E$  such that  $\Theta_{E,h_1}$  is dual Nakano positive !!

## Possible choices of the right hand side

One still has the freedom of adjusting  $f_t$ ,  $\omega_t$  and  $g_t$  in the general setup. There are in fact many possibilities:

### Proposition

Let  $(E, H_0)$  be a smooth Hermitian holomorphic vector bundle such that  $E$  is ample and  $\omega_0 = \Theta_{\det E, \det H_0} > 0$ . Then the system of determinantal and trace free equations

$$(D) \quad \det_{T_X \otimes E^*} ({}^T\Theta_{E,h_t} + (1-t)\alpha\omega_0 \otimes \text{Id}_{E^*})^{1/r} = F(t, z, h_t, D_z h_t)$$

$$(T) \quad \omega_0^{n-1} \wedge \Theta_{E,h_t}^\circ = G(t, z, h_t, D_z h_t, D_z^2 h_t) \in \text{Herm}^\circ(E, E)$$

(where  $F > 0$ ), is a well determined system of PDEs.

It is **elliptic** whenever the symbol  $\eta_h$  of the linearized operator  $u \mapsto DG_{D^2 h}(t, z, h, Dh, D^2 h) \cdot D^2 u$  has an Hilbert-Schmidt norm

$$\sup_{\xi \in T_X^*, |\xi|_{\omega_0} = 1} \|\eta_{h_t}(\xi)\|_{h_t} \leq (r^2 + 1)^{-1/2} n^{-1}$$

for any metric  $h_t$  involved, e.g. if  $G$  does not depend on  $D^2 h$ .



# Proof of the ellipticity

The (long, computational) proof consists of analyzing the linearized system of equations, starting from the curvature tensor formula

$$\Theta_{E,h} = i\bar{\partial}(h^{-1}\partial h) = i\bar{\partial}(\tilde{h}^{-1}\partial_{H_0}\tilde{h}),$$

where  $\partial_{H_0}s = H_0^{-1}\partial(H_0s)$  is the  $(1,0)$ -component of the Chern connection on  $\text{Hom}(E, E)$  associated with  $H_0$  on  $E$ .

Let us recall that the ellipticity of an operator

$$P : C^\infty(V) \rightarrow C^\infty(W), \quad f \mapsto P(f) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha f(x)$$

means the invertibility of the principal symbol

$$\sigma_P(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \in \text{Hom}(V, W)$$

whenever  $0 \neq \xi \in T_{X,x}^*$ .

For instance, on the torus  $\mathbb{R}^n/\mathbb{Z}^n$ ,  $f \mapsto P_\lambda(f) = -\Delta f + \lambda f$  has an invertible symbol  $\sigma_{P_\lambda}(x, \xi) = -|\xi|^2$ , but  $P_\lambda$  is invertible only for  $\lambda > 0$ .

## A more specific choice of the right hand side

### Theorem

The elliptic differential system defined by

$$\det_{T_X \otimes E^*} \left( {}^T \Theta_{E,h} + (1-t)\alpha\omega_0 \otimes \text{Id}_{E^*} \right)^{1/r} = \left( \frac{\det H_0(z)}{\det h_t(z)} \right)^\lambda a_0(z),$$

$$\omega_0^{n-1} \wedge \Theta_{E^\circ, h} = -\varepsilon \left( \frac{\det H_0(z)}{\det h_t(z)} \right)^\mu (\log \tilde{h}^\circ) \omega_0^n$$

possesses an **invertible elliptic linearization** for  $\varepsilon \geq \varepsilon_0(h_t)$  and  $\lambda \geq \lambda_0(h_t)(1 + \mu^2)$ , with  $\varepsilon_0(h_t)$  and  $\lambda_0(h_t)$  large enough.

### Corollary

Under the above conditions, starting from the Uhlenbeck-Yau solution  $h_0$  such that  $\det h_0 = \det H_0$  at  $t = 0$ , the PDE system **still has a solution** for  $t \in [0, t_0]$  and  $t_0 > 0$  small. (What for  $t_0 = 1$  ?)

Here, the proof consists of analyzing the **total symbol** of the linearized operator, and the rest is just linear algebra.

If  $E \rightarrow X$  is an ample vector bundle of rank  $r$  that is dual Nakano positive, one can introduce its **Monge-Ampère volume** to be

$$\text{MAVol}(E) = \sup_h \int_X \det_{T_X \otimes E^*} \left( (2\pi)^{-1} T \Theta_{E,h} \right)^{1/r},$$

where the supremum is taken over all smooth metrics  $h$  on  $E$  such that  $T \Theta_{E,h}$  is Nakano positive.

This supremum is always finite, and in fact

## Proposition

For any dual Nakano positive vector bundle  $E$ , one has

$$\text{MAVol}(E) \leq r^{-n} c_1(E)^n.$$

Taking  $\omega_0 = \Theta_{\det E}$ , the proof is a consequence of the inequality  $(\prod \lambda_j)^{1/nr} \leq \frac{1}{nr} \sum \lambda_j$  between geometric and arithmetic means, for the eigenvalues  $\lambda_j$  of  $(2\pi)^{-1} T \Theta_{E,h}$ , after raising to power  $n$ .

## Concluding remarks

- Siarhei Finski (PostDoc at Institut Fourier right now) has observed that the equality holds iff  $E$  is **projectively flat**.
- In the split case  $E = \bigoplus_{1 \leq j \leq r} E_j$  and  $h = \bigoplus_{1 \leq j \leq r} h_j$ , the inequality reads

$$\left( \prod_{1 \leq j \leq r} c_1(E_j)^n \right)^{1/r} \leq r^{-n} c_1(E)^n,$$

with equality iff  $c_1(E_1) = \dots = c_1(E_r)$ .

- In the split case, it seems natural to conjecture that

$$\text{MAVol}(E) = \left( \prod_{1 \leq j \leq r} c_1(E_j)^n \right)^{1/r},$$

i.e. that the supremum is reached for split metrics  $h = \bigoplus h_j$ .

- The Euler-Lagrange equation for the maximizer is 4th order.

# Thank you for your attention

