Shif Berhanu and Ming Xiao virtual conference on Zoom: Tuesday Aug 18 2020 until Friday August 21.

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9-9.50 am EST (15-15.50 Norway time)

Title: APPLICATION OF THE AHLFORS 5 ISLAND THEOREM IN COMPLEX DIMENSION 2

Abstract: The function $f(z) = z^k$ has the following property on the unit circle: The distance d(f(p), f(q)) = kd(p, q) so is multiplied by k for nearby points p, q. We say that f has entropy log k. In general a polynomial f(z) of degree k has entropy log k. Going to two dimensions, Smillie proved in 1990 that the Henon map F(z;w) = (f(z) + w, z) has entropy log k if f(z) is a polynomial of degree k. It is natural to think then that if f(z) is an entire transcendental function, then the entropy of F should be infinite. Indeed this is the case. The key tool is the Ahlfors 5 Island Theorem. This is work in progress together with Leandro Arosio, Anna Miriam Benini and Han Peters.

Contents

1. Introduction

Intoduction about complex dynamics. Basic example: $f(z) = z^2$. Dynamics is the study of iterations $f^{\circ n}(z) = z^{2^n}$. Where is the family of iterates well behaved, i.e. a normal family and where is it not. There are three sets: $|z| < 1$. $f^n \to 0$, $|z| > 1$. $f^n(z) \to \infty$. Both normal. $|z| = 1$. Not normal Terminology: $|z| < 1, |z| > 1$ Fatou set. $|z| = 1$ Julia set.

Why did people decide to study complex dynamics: Historicallythere are two sources, Newtons method and Celestial dynamics. These led to rational functions or polynomials.

Later this has been extended to more general complex manifolds.

Motivations:
For Newtons method the motivation was to find a way to approximate roots of polynomials, something with many applications. For celestial mechanics: Give a phenomenological understanding: Which phenomena are possible and this is in a situation where one can rely on a huge body of complex analysis. Note that for rigorous results in real life, this is anyways never possible. Even the three body problem cannot be done precisely.

Two directions: Study better the Fatou set and better the Julia set. For example for the Newton method. How many times should you iterate to get a given accuracy. The other direction is the Julia set, for example entropy.

The concept of entropy comes from physics. If X is a system and $F : X \to X$ is given. A point $x \in X$ is a possible state of the system and $F(x)$ is the state in the next moment. Then the entropy is a continuous function $g : X \to \mathbb{R}$ which is increasing, i.e. $g(F(x)) \geq g(x)$.

In our case $X = \mathbb{C}$ or $\mathbb{C}^2$ and $F : X \to X$ is a holomorphic map. In the case $F(z) = z^2$, we see that if p is a periodic point, $F^{\circ n}(p) = p$ then necessarily $g(F^m(p)) = g(p)$ for all m. This implies that $g = c$, some constant c on the unit circle. For p not on the unit circle g will increase and reaches a maximum at the origin and another at infinity, $g(0) > g_{\{|z|=1|\}}$. Both the attracting fixed points 0 and infinity are equilibrium states, as well as the points on the unit circle.

Question: What is the value of the entropy on the Julia set (or at the attracting fixed points?) Apparently, in thermodynamics, where entropy comes from, the value of the entropy is not important, it is the change in entropy that is important.

So the value of the entropy on the Julia set is a non issue. Nevertheless, entropy on equiblibrium states was introduced elsewhere, first in information theory. Then motivated by formulas used in information theory, researchers in the Soviet Union introduced entropy in dynamical systems.

This gave a value of the entropy on the unit circle, namely $\log 2$. Also it gives the value 0 for the origin.

There are two well developed directions iin complex dynamics that I will mention here. Let $f : \mathbb{C} \to \mathbb{C}$.
1. One dimensional entire functions
2. Polynomial Henon maps $H(z, w) = (f(z) + \delta w, z)$
Our project is to combine these two approaches in order to begin a study of dynamics of automorphisms in $\mathbb{C}^n$. We investigate $H(z, w) = (f(z) + \delta w, z)$ where f is entire (transcendental Henon maps).

ENTROPY:

For maps acting on compact spaces the concept of topological entropy has been introduced in 1965 (Adler- Konheim-McAndrew).

Definition 1.1 (Definition of topological entropy for compact sets)**.** Let $f : X \to X$ be a continuous self-map of a compact metric space (X, d). Let $n \in \mathbb{N}$ and $\delta > 0$. A set $E \subset X$ is called (n, δ)-*separated* if for any $z \neq w \in E$ there exists $k \leq n-1$ such that $d(f^k(z), f^k(w)) > \delta$. Let $K(n, \delta)$ be the maximal cardinality of an (n, δ)-separated set. Then the *topological entropy* $E(X, f)$ is defined as

$$E(X, f) := \sup_{\delta > 0} \left\{ \limsup_{n \to \infty} \frac{1}{n} \log K(n, \delta) \right\}.$$

In the literature there are several non-equivalent natural generalizations for the definition of topological entropy on non-compact spaces. We will use the definition introduced by Canovas and Rodríguez (2005).

Definition 1.2. Let $f : Y \to Y$ be a continuous self-map of a metric space (Y, d). Then the topological entropy $E(Y, f)$ is defined as the supremum of $E(X, f)$ over all compact subsets $X \subset Y$ for which $f(X) \subset Y$.

THE APPROACH:

Suppose you have k disjoint closed discs, $D_1, \ldots, D_k$. Let $U = \cup U_i$ and suppose that your map f has a very expansive property: $f(D_i) \supset U$ for all i. Fix an integer n and take any list of n of the D_i: $D_{i_1}, \ldots, D_{i_n}$. Then one can find a point $p_1 \in D_{i_1}$ so that $p_2 = f(p_1) \in D_{i_2}, ..., p_n = f^{n-1}(p) \in D_{i_n}$. This gives rise to k^n well separated orbits. So this gives an entropy $\frac{\log k^n}{n} = \log k$.

The collection of these orbits show that the entropy of the map f is at least $\log k$.

This method was used by Marcus Wendt (2005), a student of Bergweiler, in his (unpublished) thesis to show infinite entropy of entire transcendental functions on $\mathbb{C}$. The main tool was the Ahlfors 5 Island Theorem.

Theorem 1.3 (Ahlfors five islands Theorem)**.** *Let $D_1, \ldots, D_5$ be Jordan domains on the Riemann sphere with pairwise disjoint closures and let $D \subset \mathbb{C}$ be a domain. Then the family of all meromorphic functions $f : D \to \hat{\mathbb{C}}$ with the property that none of the D_j has a univalent preimage in D is normal.*

One has the following version:

Corollary 1.4. *Let $D_1, \ldots, D_k$ with $k \geq 3$ be bounded Jordan domains in $\mathbb{C}$ with pairwise disjoint closures and let $D \subset \mathbb{C}$ be a domain. Let $\mathcal{F}$ be a family of holomorphic functions on D which is* **not** *normal in D. Then there is an $f \in \mathcal{F}$ so that for all but at most 2 values of j, D_j has a univalent preimage in D.*

CONJUGACY INVARIANCE:

An important concept in dynamics is conjugacy invariance. It actually only means that what you study is independent on the choice of coordinates. For example, entropy is conjugacy invariant. For us, this is very important and was exploited in Wendts work. More precisely:

Let $f : \mathbb{C}(z) \to \mathbb{C}(z)$ be a holomorphic function. The dynamics of f is the study of iterations $f^{\circ n}$. The map $L(z) = w$ given by $w = z/n$ is a change of coordinates. If we calculate f in these coordinates, we get the map $g_n(w)$ where $L \circ f = g \circ L$. So $f^k = L^{-1} \circ g_n^k \circ L$.

Similarly the Henon map $F(z,w) = (f(z) + aw, z)$ is conjugate to the map $G_n(z,w) = (f_n(z) + aw, z)$ under the coordinate change $L(z,w) = (z/n, w/n)$.

The connection to Ahlfors comes from exploiting the normality or lack there of for the family of entire functions $f_n(z)$. So the idea is to use the properties that comes from the Ahlfors theorem to a suitable f_n for some large enough n.

NORMALITY PROPERTIES OF THE SEQUENCE f_n

If we fix an open set $U \subset \mathbb{C}$, we are used to questions like whether a given sequence of analytic functions $g_n : U \to \mathbb{C}$ is normal or not., i.e. whether one has subsequences which converge uniformly on compact sets to an analytic function or to infinity. For our purpose, (to use Ahlfors) we need something slightly different.

The twist is to use quasinormality. It turns out then that there are two very different lines of proof, depending on whether the f_n are quasi-normal or not (on suitable sets U).

We state the definition of quasinormality.

Definition 1.5. Let $\Omega \subset \mathbb{C}$ be a domain. A family $\mathcal{F}$ of holomorphic functions on Ω is *quasi-normal* if for every sequence (f_n) of functions in Ω there exists a finite set $Q \subset \Omega$ and a subsequence (f_{n_k}) of (f_n) which converges uniformly on compact subsets of $\Omega \setminus Q$.

Conversely:

Proposition 1.6. *Let $\Omega \subset \mathbb{C}$ be a domain and let $\mathcal{F}$ be a* **not quasi-normal family** *of holomorphic functions $\Omega \to \mathbb{C}$. Then there exists a sequence $(f_n) \subset \mathcal{F}$ and an infinite subset $Q = (x_j)_{j \geq 1} \subset \Omega$ such that no subsequence of (f_n) converges uniformly in any neighborhood of any x_j.*

2. The Quasinormal case

In this section we prove the following result:

Theorem 2.1. *Let $F : (z, w) \mapsto (f(z) - \delta w, z)$ be a transcendental Hénon map, and suppose that the transcendental functions defined by $f_n(z) = f(nz)/n$ form a* **quasi-normal** *family. Then F has infinite entropy.*

There are two steps:
(1) Show that f behaves on compact sets like a polynomial of degree d for arbitrarily large degree.
(2) Rely on the generalization by Dujardin 2004 of Smille (1990) entropy results for polynomial Henon maps

Polynomial-like maps and one-dimensional lemmas

The proof of Theorem 2.1 uses the notion of polynomial-like maps, Douady-Hubbard 1985.

Definition 2.2 (Polynomial-like maps)**.** A polynomial-like map of degree d is a branched holomorphic covering of degree d from a Jordan domain U to a Jordan domain U', with U compactly contained in U'.

It is well known that polynomial-like maps of degree d have entropy exactly $\log d$.

For any $r \in \mathbb{R}$ let us denote by $\mathbb{D}_r$ the Euclidean disk of radius r centered at 0. Let f be entire transcendental and let $\mathcal{F}$ be the family of rescalings $f_n(z) = f(nz)/n$. Assume that $\mathcal{F}$ is quasinormal. Then there is a subsequence (f_{n_k}) of (f_n) and a finite set Q such that (f_{n_k}) converges uniformly on compact sets of $\mathbb{C} \setminus Q$.

Lemma 2.3. *The set Q contains the origin, and there exists $0 < s < 1$ such that $f_{n_k} \to \infty$ uniformly on compact subsets of $\mathbb{D}_s \setminus \{0\}$.*

Proof. Observe first that for all radius $r > 0$, any subsequence of (f_n) is unbounded in the circle $\partial_r\mathbb{D}$. Indeed, for any n we have that $f_n(\mathbb{D}_{\frac{1}{\sqrt{n}}}) = f(\mathbb{D}_{\sqrt{n}})/n$, and the maximum modulus of a transcendental function on a disk of radius r grows faster than r^2.

We claim that (f_{n_k}) does not converge uniformly in a neighborhood of 0, so in particular, $0 \in Q$. Indeed, $f_{n_k}(0) = \frac{f(0)}{n_k} \to 0$ as $n_k \to \infty$, while (f_{n_k}) is unbounded in any neighborhood of 0. Since Q is finite we can find s such that $f_{n_k} \to g$ uniformly on compact subsets of $\mathbb{D}_s \setminus \{0\}$, with $g : \mathbb{D}_s \setminus \{0\} \to \mathbb{C}$ or $g = \infty$. Since (f_{n_k}) is unbounded in any circle $\partial_r\mathbb{D}$ we obtain $g = \infty$. □

Proposition 2.4. *Let $s, (f_{n_k})$ be as in Lemma 2.3. Let $0 < r < s < 1 < R$, and for k sufficiently large let U_k be the connected component of $f_{n_k}^{-1}(\mathbb{D}_R)$ containing 0. Then there exists $k_0 \in \mathbb{N}$ such that for $k > k_0$ we have*

(1) *$|f_{n_k}(z)| > R$ for every $z \in \partial\mathbb{D}_r$.*
(2) *The component U_k is compactly contained in $\mathbb{D}_r$.*
(3) *$f_{n_k} : U_k \to \mathbb{D}_R$ is polynomial-like of degree $d_k \to \infty$.*

The reason for (3) is that $f^{-1}(p)$ is usually infinite for transcendental functions.

Corollary 2.5. *Since the entropy of polynomial-like maps of degree d equals $\log d$, it follows that if $\mathcal{F}$ is quasinormal then f has infinite topological entropy.*

Henon-like maps and Proof of Theorem 2.1

The following results and definitions are from Dujardin 2004.

Let $\Delta = \mathbb{D}_{r_1} \times \mathbb{D}_{r_2}$ be a bidisk, $\partial_v(\Delta), \partial_h(\Delta)$ denote its vertical and horizontal boundary respectively. The following definition of Henon-like maps is in Dujardin04.

Definition 2.6. An injective holomorphic map H defined in a neighborhood of Δ is called *Hénon-like* if

(1) $H(\Delta) \cap \Delta \neq \emptyset$;
(2) $H(\partial_v(\Delta)) \cap \overline{\Delta} = \emptyset$;
(3) $H(\overline{\Delta}) \cap \partial\Delta \subset \partial_v(\Delta)$.

Let $\pi_z, \pi_w : \mathbb{C}^2 \to \mathbb{C}$ denote the projection to the z and to the w axis respectively.

Following Dujardin04 we have

Definition 2.7. Let H be a Henon-like map in Δ and let L_h be any horizontal line intersecting Δ. The *degree* of H is the degree of the branched covering

$$\pi_z \circ H : (H^{-1}\Delta \cap \Delta) \cap L_h \to U. \tag{2.1}$$

By Dujardin04, the map in (2.1) is proper, so the degree is well defined, and it is independent of the chosen horizontal line.

Theorem 2.8. *Let H be a Hénon-like map of degree d. The topological entropy of H is $\log d$.*

Proof of Theorem 2.1. Let $F_n(z,w) := (f_n(z) - \delta w, z)$. Recall that for each n, the maps F_n are topologically conjugate to $F = (f(z) - \delta w, z)$ via the map $(z,w) \mapsto (nz, nw)$.

In view of the fact that entropy is a topological invariant, it is enough to find a sequence (F_{n_k}) and a sequence of polydisks Δ_k on which F_{n_k} is Hénon-like of degree $d_k \to \infty$.

□

3. The Non-Quasinormal case, via AHLFORS

Theorem 3.1. *Let $F : (z,w) \mapsto (f(z) - \delta w, z)$ be a transcendental Hénon map, and suppose that the transcendental functions defined by $f_n(z) = f(nz)/n$ form a* **non quasi-normal** *family. Then F has infinite entropy.*

Assume that the family (f_n) is not quasinormal.

Let (f_{n_k}) be the subsequence of (f_n) given by Proposition 1.6 and let $Q = (x_j)_{j\geq 1}$ be the associated infinite set. Fix k. Let $R > 0$ be such that the closures of the disks $\mathbb{D}_R(x_j)$, for $j = 1, \ldots, k$ are pairwise disjoint. Next define $0 < r < R$ such that $|\delta| r < R - r$. Recall that no subsequence of (f_{n_k}) is normal in any of the k disks $\mathbb{D}_r(x_j)$, $j = 1, \ldots, k$.

Lemma 3.2. *For a given n_k, and for $i, \ell \in \{1, \ldots, k\}$ let*

$$J(i, \ell) := \{j \in \{1, \ldots, k\} : \mathbb{D}_R(x_j + \delta x_\ell) \text{ admits a biholomorphic preimage under } f_{n_k} \text{ in } \mathbb{D}_r(x_i)\}.$$

Then there exists n_k such that $\#(J(i, \ell)) \geq k - 2$ for every $i, \ell \in \{1, \ldots, k\}$.

Note that the term δx_ℓ comes from the problem that the first component $f(z) + \delta w$ has a disturbance from the δw term.

In what follows we denote the map f_{n_k} given by the previous lemma simply as f_n. We will consider the dynamics of the Hénon map $F_n(z, w) := (f_n(z) - \delta w, z)$, which is linearly conjugate to F.

Definition 3.3. Let i, ℓ both lie in $\{1, \ldots, k\}$. A holomorphic disk D is called an (i, ℓ)-disk if

- it is a holomorphic graph over $\mathbb{D}_r(x_i)$, that is D can be parametrized as $(z, w(z))$ with $w(z)$ holomorphic in $\mathbb{D}_r(x_i)$;
- $\pi_w(D) \subset \mathbb{D}_r(x_\ell)$, where π_w is the projection to the second coordinate.

Lemma 3.4. *Let $i, \ell \in \{1, \ldots, k\}$. Then for all $j \in J(i, \ell)$ and for all (i, ℓ)-disk D there exists a holomorphic disk $V \subset D$ for which $F_n(V)$ is a (j, i)-disk.*

We conclude the proof of non quasi-normal case by showing that Lemma 3.4 implies that the topological entropy of F_n is at least $\log(k - 2)$.

4. Periodic cycles of arbitrary order

We continue to a consider transcendental Hénon map F of the form

$$(z, w) \mapsto (f(z) - \delta w, z).$$

Transcendental Henon maps might not have any fixed point, nor any peridic points of order 2.

Theorem 4.1. *A transcendental Hénon map has infinitely many periodic cycles of any order $N \geq 3$.*

5. Arbitrary Growth of entropy

In Dujardin04, he constructed transcendental Hénon maps with infinite entropy by letting $f(z)$ be an entire function which, on suitable disks D_i, is well approximated by polynomials of some degree $d_i \to \infty$, and to deduce that the corresponding Hénon map is Hénon-like on the bidiscs $D_i \times D_i$ of the same degree d_i. It follows that the Hénon map has topological entropy at least $\log d_i \to \infty$.

The rate of the growth of entropy is then given by the relation between d_i and the radii of the disks D_i.

In this section we show that the entropy of *lacunary* power series, i.e. power series with mostly vanishing coefficients, can grow at any prescribed rate. We will first prove the statement for entire functions in one variable:

Theorem 5.1. *Let $h(R)$ be a continuous positive increasing function $h : [0,\infty) \to [0,\infty)$ with $h(0) = 0$ and $\lim_{R\to\infty} h(R) = \infty$. Then there exists an entire function $f(z)$ and a sequence of radii $R_j \nearrow \infty$ so that the topological entropy of f on $\{|z| \leq R_j\}$ equals $h(R_j)$.*

Then this can be applied to the Henon maps on $\Delta_{R_j} \times \Delta_{R_j}$.