Complexity of holomorphic maps from the complex unit ball to classical domains

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Abstract

We study the complexity of holomorphic isometries and proper maps from the complex unit ball to type IV classical domains. We investigate on degree estimates of holomorphic isometries and holomorphic maps with minimum target dimension. We also construct a real-parameter family of mutually inequivalent holomorphic isometries from the unit ball to type IV domains. We also provide examples of non-isometric proper holomorphic maps from the complex unit ball to classical domains.

1 Introduction

The motivation of this paper is twofold: the study of both isometries and proper maps between bounded symmetric space. Let D, Ω be bounded symmetric domains equipped with the Bergman metrics ω_D, ω_Ω respectively. A holomorpic map $F : D \to \Omega$ is called isometric if $F^*\omega_\Omega = \lambda\omega_D$ for a positive constant λ. Holomorphic isometries between bounded symmetric domains was first studied by Clozel-Ullmo [CU] and later systematically studied by Mok. He [M5] proved a holomorphic isometry F is totally geodesic when D is irreducible and of rank at least 2. When D is the complex unit ball of dimension at least 2 and Ω is the product of complex unit balls, F is also totally geodesic [YZ]. Much less is understood otherwise. When D is the complex unit ball and Ω is an irreducible bounded symmetric domain of rank at least 2, a surprising non-totally geodesic phenomenon was discovered by Mok [M6]. More precisely, for each irreducible Ω, there exists a positive integer n_Ω such that $\mathbb{B}^{n_\Omega}$ admits a non-totally geodesic holomorphic isometry to Ω [M6]. Moreover, if $\mathbb{B}^n$ admits a holomorphic isometry to Ω, then $n \leq n_\Omega$ [M6]. More recently, Chan-Mok characterize the image of the complex unit ball in Ω under the holomorphic isometry [CM]. When Ω is the type IV classical domain, similar classification results are obtained independently in [CM], [UWZ], [XY]. The related problems

on holomorphic isometries or holomorphic maps preserving invariant forms in Hermitian symmetric spaces are considered in [C], [M2], [Ng1], [Ng2], [M5], [MN1], [MN2], [HY1], [HY2], [Ch], [Eb2] [FHX], [Yu], et al.

Proper holomorphic maps between bounded symmetric domains have also been a central subject in analysis and geometry of several complex variables. Let $F : D \to \Omega$ be a proper holomorphic map. When D is of rank at least 2, many rigidity and non-rigidity results have been obtained in [TH], [Ts], [Tu1], [Tu2], [M3], [KZ1], [KZ2], [Ng3], et al. When $D = \mathbb{B}^n, \Omega = \mathbb{B}^N$, the rigidity and complexity of proper holomorphic maps remains a rather active problem in several complex variables. Since Poincaré's pioneer work [P], many experts made fundamental progresses along the line (See [Al], [L], [Fo], [St], [W], [Fa], [CS], [D1], [Hu1], [Eb1] and many references therein). Roughly speaking, if the difference between n, N is small, F can be fully classified with additional assumptions on the boundary regularity of the map (cf. [Al], [Fa], [Hu1], [HJ], [Ha]). In general, for arbitrary n, N, the moduli space of proper maps from $\mathbb{B}^n$ to $\mathbb{B}^N$ is rather large (cf. [CD], [DL2]).

A gap phenomenon was then discovered in [HJX] and later a gap conjecture was formulated by Huang-Ji-Yin [HJY1] (see [HJY2] as well). A proper holomorphic map $F : \mathbb{B}^n \to \mathbb{B}^N$ is called minimum if it cannot be reduced to a map $(G, 0)$ where G has smaller target dimension modulo automorphisms of $\mathbb{B}^n$ and $\mathbb{B}^N$. The gap conjecture predicts precisely the intervals of N where there are no minimum holomorphic proper maps from $\mathbb{B}^n$ to $\mathbb{B}^N$. They also showed that when N is not in these intervals, then there is always a minimum monomial proper map from $\mathbb{B}^n$ to $\mathbb{B}^N$. These intervals terminate when the target dimension gets too large. (See als the work by D'Angelo and Lebl [DL1]).

The authors showed in a recent paper [XY] that when $D = \mathbb{B}^n, \Omega = D_m^{IV}$ and the codimension is small, any proper holomorphic map is indeed an isometry with additional boundary regularity assumptions. In this paper, we continue to study the complexity of holomorphic proper and isometric maps from the complex unit ball to an irreducible classical domain. Motivated by the gap conjecture mentioned above, we study holomorphic proper (resp. isometric) maps $F : \mathbb{B}^n \to D_m^{IV}$ with minnimum target dimension (See Section 3 for the precise definition). We show in Section 3 that holomorphic isometries from $\mathbb{B}^n$ to D_m^{IV} are *not* minimum if $m \geq 2n + 3$. We also illustrate this result is optimal by constructing minimum holomorphic isometries from $\mathbb{B}^n$ to D_m^{IV} for each $n + 2 \leq m \leq 2n + 2$. On the other hand, we prove that there always exist minimum proper holomorphic maps from $\mathbb{B}^n$ to D_m^{IV} for any $m \geq n + 1$. In Section 4, we prove that there exists a real-parameter family of mutually inequivalent minimum holomorphic isometries from $\mathbb{B}^n$ to D_m^{IV} if $n + 2 \leq m \leq 2n + 2$. Section 5 is devoted to establishing degree estimates for holomorphic isometric maps from $\mathbb{B}^n$ to D_m^{IV}. In Section 6, we construct non-totally geodesic proper holomorphic maps from $\mathbb{B}^n$ to Ω where Ω is any of the four types of classical domains. These maps can be non-isometric when n is small while they become isometries when $n = n_\Omega$. Interestingly, these examples further provide polynomial proper holomorphic maps from $\mathbb{B}^n$ to Ω that answer a question of Mok (see also the independent work of Chan-Mok [CM]).

A large part of the paper was finished in the December of 2015, before we learned of many

interesting results on holomorphic isometries from $\mathbb{B}^n$ to D_m^{IV} proved by Chan-Mok [CM]. By combining the theorems of Chan-Mok [CM] and ours [XY] (see the similar result in [UWZ] as well), it is clear that any holomorphic isometry $F : \mathbb{B}^n \to D_m^{IV}$ can be written in the following form $F = \varphi \circ f \circ \tau \circ i \circ \sigma$, where $\sigma \in Aut(\mathbb{B}^n), \tau \in Aut(\mathbb{B}^{m-1}), \varphi \in Aut(D_m^{IV})$, i is the standard linear embedding from $\mathbb{B}^n$ to $\mathbb{B}^{m-1}$ and f is either R_{m-1}^{IV} or I_{m-1}^{IV}, which are defined in Section 6.4. The study on minimality and inequivalent families of holomorphic isometries from the complex unit ball to the type IV classical domains in Section 3 and Section 4 is motivated by the analogue study for proper holomorphic maps between balls.

2 Preliminaries

Irreducible bounded symmetric domains are realized as Cartan's four types of domains and two exceptional cases (cf. [H2] [M1]). Assume $q \geq p$ and let $M(p, q; \mathbb{C})$ denote the space of $p \times q$ matrices with entries of complex numbers. The type I domain is defined as

$$D_{p,q}^I = \{Z \in M(p, q; \mathbb{C}) : I_p - Z\overline{Z}^t > 0\}.$$

In particular, when $p = 1$, the type I domain is the complex unit ball $\mathbb{B}^q = \{z = (z_1, \cdots, z_q) \in \mathbb{C}^q : |z|^2 < 1\}$ in $\mathbb{C}^q$. The type II and type III domains are submanifolds of $D_{n,n}^I$ defined as

$$D_n^{II} = \{Z \in D_{n,n}^I : Z = -Z^t\}$$

and

$$D_n^{III} = \{Z \in D_{n,n}^I : Z = Z^t\}.$$

The type IV domain is defined as

$$D_n^{IV} = \{Z = (z_1, \cdots, z_n) \in \mathbb{C}^n : Z\overline{Z}^t < 2 \text{ and } 1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2 > 0\}.$$

Let Ω be an irreducible classical domain. The Bergman kernel function $K_\Omega(Z, \bar{Z})$ is explicitly given by

$$\begin{aligned}
K_\Omega(Z, \bar{Z}) &= c_I \left(\det(I_p - Z\overline{Z}^t)\right)^{-(p+q)} && \text{when } \Omega = D_{p,q}^I; \\
K_\Omega(Z, \bar{Z}) &= c_{II} \left(\det(I_n - Z\overline{Z}^t)\right)^{-(n-1)} && \text{when } \Omega = D_n^{II}; \\
K_\Omega(Z, \bar{Z}) &= c_{III} \left(\det(I_n - Z\overline{Z}^t)\right)^{-(n+1)} && \text{when } \Omega = D_n^{III}; \\
K_\Omega(Z, \bar{Z}) &= c_{IV} \left(1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2\right)^{-n} && \text{when } \Omega = D_n^{IV},
\end{aligned} \tag{1}$$

where c_* are positive constants depending on n and the type of Ω (cf. [H2] [M1]). The Bergman metric

$$\omega_\Omega(Z) := \sqrt{-1}\partial\bar{\partial}\log K_\Omega(Z, \bar{Z})$$

on Ω is Kähler-Einstein as the Bergman kernel function is invariant under holomorphic automorphisms. Note that the standard linear embedding $L(Z) = Z$ from D_n^{II} or D_n^{III} into $D_{n,n}^I$ is a totally geodesically holomorphic isometric embedding with respect to Bergman metrics with isometric constant $\frac{2n}{n-1}$ or $\frac{2n}{n+1}$ respectively.

Let S be the Hermitian symmetric space of compact type dual to Ω and $\delta \in H^2(S, \mathbb{Z})$ be the positive generator. It is well-known that the first Chern class $c_1(S) = (p+q)\delta, 2(n-1)\delta, (n+1)\delta$ or $n\delta$, When $\Omega = D_{p,q}^I, D_n^{II}, D_n^{III}, D_n^{IV}$ respectively. Therefore, it follows from Mok's theorem in [M6] that $n_\Omega = p + q - 1, 2n - 3, n$ or $n - 1$ when the classical domain $\Omega = D_{p,q}^I, D_n^{II}, D_n^{III}$ or D_n^{IV} respectively.

We now describe the holomorphic automorphism group action on D_m^{IV} in terms of the Borel embedding (cf. [H1] [M1]). The hyperquadric $\mathbb{Q}^m$, the compact dual of D_n^{IV} is defined by $\mathbb{Q}^m := \{[z_1, \cdots, z_{m+2}] \in \mathbb{P}^{m+1} | z_1^2 + \cdots + z_m^2 = z_{m+1}^2 + z_{m+2}^2\}$. The Borel embedding $D_m^{IV} \subset \mathbb{Q}^m \subset \mathbb{P}^{m+1}$ is given by

$$Z = (z_1, \cdots, z_m) \to \left[z_1, \cdots, z_m, \frac{1+\frac{1}{2}ZZ^t}{\sqrt{2}}, \frac{1-\frac{1}{2}ZZ^t}{\sqrt{-2}}\right].$$

The holomorphic automorphism group of D_m^{IV} is given by

$$\mathrm{Aut}(D_m^{IV}) = \left\{\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in O(m, 2, \mathbb{R}) | \det(D) > 0\right\},$$

where $A \in M(m, m, \mathbb{R}), B \in M(m, 2, \mathbb{R}), C \in M(2, m, \mathbb{R}), D \in M(2, 2, \mathbb{R})$. The automorphism group action is given in the following explicit way. Let $Z = (z_1, \cdots, z_m) \in D_m^{IV}$ and $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Aut}(D_n^{IV})$. Write $Z' = \left(\frac{1+\frac{1}{2}ZZ^t}{\sqrt{2}}, \frac{1-\frac{1}{2}ZZ^t}{\sqrt{-2}}\right)$. Then the action of T on D_m^{IV} is given by

$$T(Z) = \frac{ZA + Z'C}{(ZB + Z'D)\left(1/\sqrt{2}, \sqrt{-1/2}\right)^t}.$$

Rephrasing in homogenous coordinates, if the holomorphic automorphism maps $Z = (z_1, \cdots, z_m) \in D_m^{IV}$ to $W = (w_1, \cdots, w_m) \in D_m^{IV}$, then there exists $T \in \mathrm{Aut}(D_m^{IV})$ such that

$$\left[w_1, \cdots, w_m, \frac{1+\frac{1}{2}WW^t}{\sqrt{2}}, \frac{1-\frac{1}{2}WW^t}{\sqrt{-2}}\right] = \left[z_1, \cdots, z_m, \frac{1+\frac{1}{2}ZZ^t}{\sqrt{2}}, \frac{1-\frac{1}{2}ZZ^t}{\sqrt{-2}}\right] \cdot T.$$

In other words, there exists nonzero $\lambda \in \mathbb{C}$, such that

$$\left(w_1, \cdots, w_m, \frac{1+\frac{1}{2}WW^t}{\sqrt{2}}, \frac{1-\frac{1}{2}WW^t}{\sqrt{-2}}\right) = \lambda \left(z_1, \cdots, z_m, \frac{1+\frac{1}{2}ZZ^t}{\sqrt{2}}, \frac{1-\frac{1}{2}ZZ^t}{\sqrt{-2}}\right) \cdot T.$$

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Note that the isotropy group K_0 at the origin is $K_0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in O(m, 2, \mathbb{R}) | \det(D) = 1 \right\} \cong$ $O(m, \mathbb{R}) \times SO(2, \mathbb{R})$.

3 Holomorphic maps from $\mathbb{B}^n$ to D_m^{IV}

In this section, we study the holomorphic maps from $\mathbb{B}^n$ to D_m^{IV} with minimum dimension in the target. We say two holomorphic maps $F_1, F_2 : \mathbb{B}^n \to D_m^{IV}$ are equivalent if $F_1 = \phi \circ F_2 \circ \psi$, where ψ, ϕ are automorphisms of $\mathbb{B}^n$ and D_m^{IV}, respectively. A holomorphic map $F : \mathbb{B}^n \to D_m^{IV}$ is called minimum if there is no holomorphic map $G : \mathbb{B}^n \to D_{m-1}^{IV}$ such that F is equivalent to $(G, 0)$.

3.1 Minimum of holomorphic isometries to Type IV domains

We study in this subsection the minimum holomorphic isometric maps from $\mathbb{B}^n$ to D_m^{IV}. We first note that it follows from Mok's theorem [M6] that any holomorphic isometry $F : \mathbb{B}^n \to D_{n+1}^{IV}$ is minimum. We prove the following theorem that there are no minimum holomorphic isometric maps from $\mathbb{B}^n$ to D_m^{IV} if $m > 2n + 2$.

Theorem 3.1. *Let $m > 2n + 2, n \geq 2$. Let $F : \mathbb{B}^n \to D_m^{IV}$ be a holomorphic isometry. Then there exists a holomorphic isometry $G : \mathbb{B}^n \to D_{2n+2}^{IV}$ such that F is equivalent to $(G, \mathbf{0})$, where $\mathbf{0}$ is a $(m - 2n - 2)$-dimensional zero row vector.*

Proof. It suffices to show that for any such F and m, there exists a holomorphic isometry $\hat{F} : \mathbb{B}^n \to D_{m-1}^{IV}$ such that F is equivalent to $(\hat{F}, \mathbf{0})$. By the isometric assumption, we have

$$F^*(\omega_{D_m^{IV}}) = \lambda \omega_{\mathbb{B}^n}, \tag{2}$$

for some positive constant λ. From Proposition 2.11 in [XY], we know that λ must be $m/(n+1)$. Write $Z = (z_1, ..., z_n)$ as the coordinates in $\mathbb{C}^n$. Write $F = (F_1, \cdots, F_m)$. By composing with the autmorphism of D_m^{IV} if necessary, we may assume $F(0) = 0$. By standard reduction, we obtain from (2),

$$1 - F\overline{F}^t + \frac{1}{4}|FF^t|^2 = 1 - Z\overline{Z}^t.$$

By a lemma of D'Angelo ([D2]), we have

$$(F_1, \cdots, F_m) = (z_1, \cdots, z_n, \frac{1}{2}\sum_{i=1}^m F_i^2(z), 0, \cdots, 0) \cdot \mathbf{V},$$

where $\mathbf{V}$ is an $m \times m$ unitary matrix. Write

$$V = \begin{bmatrix} \mathbf{v_1} \\ \cdots \\ \mathbf{v_m} \end{bmatrix},$$

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where $\mathbf{v_i}$ is a m-dimensional row vector for $1 \le i \le m$. Then we have

$$\left(z_1, \cdots, z_n, \frac{1}{2} \sum_{i=1}^{m} F_i^2(z) \right) \cdot \begin{bmatrix} \mathbf{v_1} \\ \cdots \\ \mathbf{v_{n+1}} \end{bmatrix} = (F_1, \cdots, F_m).$$

Claim: $\{F_1, \cdots, F_m\}$ is a linearly dependent set over real number field. In other words, there exist $\lambda_1, \cdots, \lambda_m \in \mathbb{R}$ not mutually zero, such that $\sum_{i=1}^{m} \lambda_i F_i \equiv 0$.

Proof of Claim: Write $\mathbf{v_i} = \mathbf{a_i} + \sqrt{-1}\mathbf{b_i}$ for $1 \le i \le n+1$. It is easy to see that there exists $\mathbf{v} = (\lambda_1, \cdots, \lambda_m)^t \in \mathbb{R}^m$ with $\mathbf{v} \ne 0$ such that

$$\begin{bmatrix} \mathbf{a_1} \\ \mathbf{b_1} \\ \cdots \\ \mathbf{a_{n+1}} \\ \mathbf{b_{n+1}} \end{bmatrix} \mathbf{v} = \mathbf{0}. \tag{3}$$

This is because

$$\text{rank} \begin{bmatrix} \mathbf{a_1} \\ \mathbf{b_1} \\ \cdots \\ \mathbf{a_{n+1}} \\ \mathbf{b_{n+1}} \end{bmatrix} \le 2(n+1) < m.$$

Then (3) implies

$$\left(z_1, \cdots, z_n, \frac{1}{2} \sum_{i=1}^{m} F_i^2(z) \right) \cdot \begin{bmatrix} \mathbf{v_1} \\ \cdots \\ \mathbf{v_{n+1}} \end{bmatrix} \cdot \mathbf{v} = \mathbf{0}.$$

This proves the claim by showing $\sum_{i=1}^{m} \lambda_i F_i \equiv 0$.

By rescaling $\mathbf{v}$ if necessary, we assume $|\mathbf{v}| = 1$. Extend $\mathbf{v}$ to an orthonormal basis $\{\mathbf{u_1}, \cdots, \mathbf{u_{m-1}}, \mathbf{v}\}$ of $\mathbb{R}^m$ and write $m \times m$ matrix $\mathbf{C} = (\mathbf{u_1}, \cdots, \mathbf{u_{m-1}}, \mathbf{v})$. Define $\hat{F} = (\hat{F}_1, \cdots, \hat{F}_m) = F \cdot \mathbf{C}$ and then $\hat{F}$ is equivalent to F. This completes the proof of the theorem because $\hat{F}_m = F \cdot \mathbf{v} = 0$. $\qquad \square$

Define $R_{n+2}, I_{n+2} : \mathbb{B}^n \to D_{n+2}^{IV}$ to be

$$R_{n+2}(z) = \left(\cos\theta_1 z_1, \sqrt{-1} \sin\theta_1 z_1, z_2, \cdots, z_{n-1}, \right.$$

$$\frac{\cos(2\theta_1)z_1^2 + \sum_{j=2}^{n-1} z_j^2 - 2z_n^2 + 2z_n}{2\sqrt{2}(1-z_n)}, \frac{\cos(2\theta_1)z_1^2 + \sum_{j=2}^{n-1} z_j^2 + 2z_n^2 - 2z_n}{2\sqrt{-2}(1-z_n)} \right),$$

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$$I_{n+2}(z) = \left(\cos\theta_1 z_1, \sqrt{-1}\sin\theta_1 z_1, z_2, \cdots, z_n, 1 - \sqrt{1 - \cos(2\theta_1)z_1^2 - \sum_{j=2}^{n} z_j^2} \right)$$

with $\theta \in (0, \pi/4]$; and one can similarly define $R_{n+k}, I_{n+k} : \mathbb{B}^n \to D_{n+k}^{IV}$ for $2 \le k \le n$:

$$R_{n+k}(z) = \left(\cos\theta_1 z_1, \sqrt{-1}\sin\theta_1 z_1, \cdots, \cos\theta_{k-1} z_{k-1}, \sqrt{-1}\sin\theta_{k-1} z_{k-1}, \right.$$

$$z_k, \cdots, z_{n-1}, \frac{\sum_{j=1}^{k-1}\cos(2\theta_j)z_j^2 + \sum_{j=k}^{n-1} z_j^2 - 2z_n^2 + 2z_n}{2\sqrt{2}(1 - z_n)}, \left. \frac{\sum_{j=1}^{k-1}\cos(2\theta_j)z_j^2 + \sum_{j=k}^{n-1} z_j^2 + 2z_n^2 - 2z_n}{2\sqrt{-2}(1 - z_n)} \right)$$

$$I_{n+k}(z) = \left(\cos\theta_1 z_1, \sqrt{-1}\sin\theta_1 z_1, \cdots, \cos\theta_{k-1} z_{k-1}, \sqrt{-1}\sin\theta_{k-1} z_{k-1}, \right.$$

$$\left. z_k, \cdots, z_n, 1 - \sqrt{1 - \sum_{j=1}^{k-1}\cos(2\theta_j)z_j^2 - \sum_{j=k}^{n} z_j^2} \right)$$

with $\theta_j \in (0, \pi/4]$ for $1 \le j \le k-1$. Here when $k = n$, the components "$z_k, \cdots, z_{n-1}$" in R_{n+k} is understood to be void. Furthermore, $R_{2n+1}, I_{2n+1} : \mathbb{B}^n \to D_{2n+1}^{IV}$ are given by

$$R_{2n+1}(z) = \left(\cos\theta_1 z_1, \sqrt{-1}\sin\theta_1 z_1, \cdots, \cos\theta_{n-1} z_{n-1}, \sqrt{-1}\sin\theta_{n-1} z_{n-1}, z_n, \right.$$

$$\left. \frac{1}{2\sqrt{2}}\left(\sum_{j=1}^{n-1}\cos(2\theta_j)z_j^2 + z_n^2 \right), \frac{-\sqrt{-1}}{2\sqrt{2}}\left(\sum_{j=1}^{n-1}\cos(2\theta_j)z_j^2 + z_n^2 \right) \right)$$

for $\theta_1, \cdots, \theta_{n-1} \in (0, \pi/4]$,

$$I_{2n+1}(z) = \left(\cos\theta_1 z_1, \sqrt{-1}\sin\theta_1 z_1, \cdots, \cos\theta_n z_n, \sqrt{-1}\sin\theta_n z_n, 1 - \sqrt{1 - \sum_{j=1}^{n}\cos(2\theta_j)z_j^2} \right)$$

for $\theta_1, \cdots, \theta_n \in (0, \pi/4]$ but not all $\theta_j = \pi/4$; and $R_{2n+2}, I_{2n+2} : \mathbb{B}^n \to D_{2n+2}^{IV}$ are given by

$$R_{2n+2}(z) = \left(\cos\theta_1 z_1, \sqrt{-1}\sin\theta_1 z_1, \cdots, \cos\theta_n z_n, \sqrt{-1}\sin\theta_n z_n, \right.$$

$$\left. \frac{1}{2\sqrt{2}}\sum_{j=1}^{n}\cos(2\theta_j)z_j^2, \frac{-\sqrt{-1}}{2\sqrt{2}}\sum_{j=1}^{n}\cos(2\theta_j)z_j^2 \right),$$

$$I_{2n+2}(z) = \left(\cos\theta_1 z_1, \sqrt{-1}\sin\theta_1 z_1, \cdots, \cos\theta_n z_n, \sqrt{-1}\sin\theta_n z_n, \right.$$

$$\frac{\cos\theta}{\cos(2\theta)}\left(1 - \sqrt{1 - \cos(2\theta)\left(\sum_{j=1}^{n}\cos(2\theta_j)z_j^2\right)}\right), \frac{\sqrt{-1}\sin\theta}{\cos(2\theta)}\left(1 - \sqrt{1 - \cos(2\theta)\left(\sum_{j=1}^{n}\cos(2\theta_j)z_j^2\right)}\right)\right)$$

for $\theta_j \in (0, \pi/4]$ but not all $\theta_j = \pi/4$ for $1 \le j \le n$ and $\theta \in (0, \pi/4)$.

Theorem 3.2. *For each $2 \le k \le n+2$, $R_{n+k}, I_{n+k} : \mathbb{B}^n \to D_{n+k}^{IV}$ are minimum holomorphic isometries.*

Proof. We will only prove the case $k = 2$ and other cases follow by similar argument. Apply the Borel embedding to embed $\mathbb{B}^n$ as an open subset of $\mathbb{P}^n$ and D_m^{IV} as an open subset of $\mathbb{Q}^m \subset \mathbb{P}^{m+1}$ and write $[z, s] = [z_1, \cdots, z_n, s]$ to denote the homogeneous coordinates in $\mathbb{P}^n$. Here recall the Borel embedding of $\mathbb{B}^n$ into $\mathbb{P}^n$ is given by

$$(z_1, \cdots, z_n) \to [z_1, \cdots, z_n, 1].$$

The Borel embedding of D_m^{IV} into $\mathbb{Q}^m \subset \mathbb{P}^{m+1}$ is as described in Section 2.

We first prove the theorem for R_{n+2}. Under the homogeneous coordinates, R_{n+2} can be identified with

$$\mathcal{R}_{n+2}(z, s) = [g_1(z, s), \cdots, g_{n+4}(z, s)]$$

from $\mathbb{P}^n$ to $\mathbb{P}^{n+3}$, where

$$g_1(z, s) = \cos\theta_1(s - z_n)z_1;$$
$$g_2(z, s) = \sqrt{-1}\sin\theta_1(s - z_n)z_1;$$
$$g_j(z, s) = (s - z_n)z_{j-1} \text{ for } 3 \le j \le n;$$
$$g_{n+1}(z, s) = \frac{\cos(2\theta_1)z_1^2 + \sum_{j=2}^{n-1} z_j^2 - 2z_n^2 + 2z_n s}{2\sqrt{2}};$$
$$g_{n+2}(z, s) = \frac{\cos(2\theta_1)z_1^2 + \sum_{j=2}^{n-1} z_j^2 + 2z_n^2 - 2z_n s}{2\sqrt{-2}};$$
$$g_{n+3}(z, s) = \frac{1}{2\sqrt{2}}\left(\cos(2\theta_1)z_1^2 + \sum_{j=2}^{n} z_j^2 + 2s^2 - 2z_n s\right);$$
$$g_{n+4}(z, s) = \frac{1}{2\sqrt{-2}}\left(-\cos(2\theta_1)z_1^2 - \sum_{j=2}^{n} z_j^2 + 2s^2 - 2z_n s\right).$$

Claim: The set $\{g_1, \cdots, g_{n+4}\}$ is linearly independent over $\mathbb{R}$ on any open subset of $\mathbb{C}^{n+1}$. Consequently, for any $\mathbf{B} \in U(n, 1)$, the set $\{\hat{g}_1, \cdots, \hat{g}_{n+4}\}$ with $\hat{g}_j = g_j((z, s) \cdot \mathbf{B})$ is linearly independent over $\mathbb{R}$.

Proof of Claim: We will just prove the first part of the claim and the second part follows easily. Let $\{a_1, \cdots, a_{n+4}\}$ be a set of real numbers such that $\sum_{j=1}^{n+4} a_j \hat{g}_j \equiv 0$. By comparing

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coefficients of $z_n z_j$ for $1 \le j \le n-1$, we know $a_j = 0$ for $1 \le j \le n$. By comparing coefficients of z_n^2, we know $a_{n+1} = a_{n+2} = 0$. By comparing coefficients of s^2, we know $a_{n+3} = a_{n+4} = 0$. This proves the claim.

Now suppose that R_{n+2} is not minimum. Namely, there exists $F : \mathbb{B}^n \to D_m^{IV}$ with $m < n+2$ such that R_{n+2} is equivalent to $(F, 0)$. More precisely, under homogeneous coordinates, there exist $\mathbf{B} \in U(n, 1)$ and $T \in \mathrm{Aut}(D_{n+2}^{IV})$ such that

$$\mathcal{R}_{n+2}((z, s) \cdot \mathbf{B}) \cdot T = [\tilde{F}(z, s), 0, \cdots], \tag{4}$$

where $\tilde{F}$ is the map obtained from F under homogeneous coordinates. By comparing the $(n + 2)$-th element in (4), we deduce a contradiction to the claim. This shows that R_{n+2} must be minimum.

The conclusion for I_{n+2} in the theorem follows from the similar argument. Under the homogeneous coordinates, I_{n+2} can be identified with

$$\mathcal{I}_{n+2}(z, s) = [h_1(z, s), \cdots, h_{n+4}(z, s)]$$

from $\mathbb{P}^n$ to $\mathbb{P}^{n+3}$, where

$$
\begin{aligned}
h_1(z, s) &= \cos\theta_1 z_1; \\
h_2(z, s) &= \sqrt{-1}\sin\theta_1 z_1; \\
h_j(z, s) &= z_{j-1} \text{ for } 3 \le j \le n+1; \\
h_{n+2}(z, s) &= s - \sqrt{s^2 - \cos(2\theta_1)z_1^2 - \sum_{j=2}^{n} z_j^2}; \\
h_{n+3}(z, s) &= \frac{1}{\sqrt{2}}\left(2s - \sqrt{s^2 - \cos(2\theta_1)z_1^2 - \sum_{j=2}^{n} z_j^2}\right); \\
h_{n+4}(z, s) &= \frac{1}{\sqrt{-2}}\sqrt{s^2 - \cos(2\theta_1)z_1^2 - \sum_{j=2}^{n} z_j^2}.
\end{aligned}
$$

Claim: The set $\{h_1, \cdots, h_{n+4}\}$ is linearly independent over $\mathbb{R}$ on any open subset of $\mathbb{C}^{n+1}$. Consequently, for any $\mathbf{B} \in U(n, 1)$, the set $\{\hat{h}_1, \cdots, \hat{h}_{n+4}\}$ with $\hat{h}_j = h_j((z, s) \cdot \mathbf{B})$ is linearly independent over $\mathbb{R}$.

The proof the claim is very similar to the previous one. Let $\{a_1, \cdots, a_{n+4}\}$ be a set of real numbers such that $\sum_{j=1}^{n+4} a_j \hat{h}_j \equiv 0$. Then one can show $a_j = 0$ for all j by comparing coefficients.

The rest proof of the theorem is also similar. Suppose that I_{n+2} is not minimum. Namely, there exists $F : \mathbb{B}^n \to D_m^{IV}$ with $m < n+2$ such that I_{n+2} is equivalent to $(F, 0)$. More precisely,

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under homogeneous coordinates, there exist $\mathbf{B} \in U(n,1)$ and $T \in \mathrm{Aut}(D_{n+2}^{IV})$ such that

$$\mathcal{I}_{n+2}((z,s) \cdot \mathbf{B}) \cdot T = (\tilde{F}(z,s), 0, \cdots), \tag{5}$$

where $\tilde{F}$ is the map obtained from F under homogeneous coordinates. By comparing the $(n+2)$-th element in (5), we deduce a contradiction to the claim. This shows that I_{n+2} must be minimum. $\qquad\square$

3.2 Minimum holomorphic proper maps to Type IV domains

We investigate minimim holomorphic proper maps from $\mathbb{B}^n$ to D_m^{IV} in this subsection. By Lemma 2.2 in [XY], there is no proper holomorphic maps from $\mathbb{B}^n$ to D_m^{IV} if $n \geq 2, m \leq n,$. The following theorem reveals a different phenomenon of proper holomorphic maps from isometries.

Theorem 3.3. *For any $m \geq n+1 \geq 2$, there is a minimum proper holomorphic map from $\mathbb{B}^n$ to D_m^{IV}.*

To establish Theorem 3.3, we first prove the following result.

Theorem 3.4. *Let $N \geq n \geq 1$. Let $F = (f_1, \cdots, f_N)$ be a minimum monomial proper map from $\mathbb{B}^n$ to $\mathbb{B}^N$, where each $f_i = c_i z^{\alpha_i}$ for some multiindex α_i and real number c_i. Define holomorpic maps from $\mathbb{B}^n$ to $\mathbb{C}^{N+k}, 1 \leq k \leq N+1$ associated to F as follows.*

$$M_F^{N+1} := \left(f_1, \cdots, f_N, 1 - \sqrt{1 - \sum_{i=1}^N f_i^2} \right),$$

and for each $2 \leq k \leq N+1$, fixing $\theta \in (0, \frac{\pi}{4})$, define

$$M_F^{N+k} := \Big(\cos\theta f_1, \sqrt{-1}\sin\theta f_1, \cdots, \cos\theta f_{k-1}, \sqrt{-1}\sin\theta f_{k-1}$$

$$f_k, \cdots, f_N, 1 - \sqrt{1 - \cos(2\theta)\sum_{j=1}^{k-1} f_j^2 - \sum_{j=k}^N f_j^2} \Big).$$

Then each $M_F^{N+k}, 1 \leq k \leq N+1$, is a minimum proper holomorphic map from $\mathbb{B}^n$ to D_{N+k}^{IV}.

Proof. We will prove only the statement for $k = 1$. The other cases can be proved by a similar argument. Write $m = N+1, H = M_F^{N+1}$. Denote the highest degree of $f_i, 1 \leq i \leq N$ by d.

By direct computation one can verify that H is a proper holomorphic map from $\mathbb{B}^n$ to D_m^{IV}. We now prove that H is minimum. For that we again apply the Borel embedding to embed $\mathbb{B}^n$ as an open subset of $\mathbb{P}^n$ and D_m^{IV} as an open subset $\mathbb{Q}^m \subset \mathbb{P}^{m+1}$. Write $[z,s] = [z_1, ..., z_n, s]$ to

denote the homogeneous coordinates in $\mathbb{P}^n$. Under the homogeneous coordinates, H is identified with

$$\mathcal{H}(z, s) = [h_1(z, s), ..., h_{m+2}(z, s)]$$

from $\mathbb{P}^n$ to $\mathbb{P}^{m+1}$, where

$$h_1(z, s) = s^d f_1\left(\frac{z}{s}\right), \quad \cdots, \quad h_N(z, s) = s^d f_N\left(\frac{z}{s}\right),$$

$$h_m(z, s) = s^d - \sqrt{s^{2d} - \sum_{i=1}^N h_i^2},$$

$$h_{m+1}(z, s) = \frac{1}{\sqrt{2}}\left(2s^d - \sqrt{s^{2d} - \sum_{i=1}^N h_i^2}\right),$$

$$h_{m+2}(z, s) = \frac{1}{\sqrt{-2}}\sqrt{s^{2d} - \sum_{i=1}^N h_i^2}.$$

Claim 3.5. *The set $\{h_1, ..., h_{m+2}\}$ is linearly independent over $\mathbb{R}$. Consequently, for any $\mathbf{B} \in U(n, 1)$, the set $\{\hat{h}_1, \cdots, \hat{h}_{m+2}\}$ with $\hat{h}_j(z, s) = h_j((z, s) \cdot \mathbf{B})$ is linearly independent over $\mathbb{R}$.*

We first note that $s^{2d} - \sum_{i=1}^N h_i^2$ is not a perfect square of a polynomial. This amounts to the following easy lemma whose proof will be left to readers.

Lemma 3.6. *Let $f_i, 1 \leq i \leq N$, be mutually distinct monomials with real coefficients. Then $1 - \sum_{i=1}^N f_i^2$ is not a perfect square of a polynomial.*

It follows from Lemma 3.6 that $\{h_1, ..., h_N, s^d, \sqrt{s^{2d} - \sum_{i=1}^N h_i^2}\}$ is linearly independent over $\mathbb{R}$. This further implies $\{h_1, ..., h_{m+2}\}$ is also linearly independent over $\mathbb{R}$. The latter part of Claim 3.5 is then an easy consequence. The rest of proof is just a copy of that of Theorem 3.2. $\qquad\square$

We recall the results about the gap conjecture on proper maps between balls. The following intervals appear in the gap conjecture for proper holomorphic maps between balls [HJY1]. For $n > 2$, let $K(n) := \max\{m \in \mathbb{Z}^+ : m(m+1)/2 < n\}$ and let $I_k := \{m \in \mathbb{Z}^+ : kn < m < (k+1)n - k(k+1)/2\}$ for $1 \leq k \leq K(n)$. The following theorem is proved by Huang-Ji-Yin (See also D'Angelo-Lebl [DL1]).

Theorem 3.7. *([HJY1]) For any $N \geq n \geq 2$ with $N \notin \cup_{k=1}^{K(n)} I_k$, there is a minimum proper monomial map from $\mathbb{B}^n$ to $\mathbb{B}^N$.*

Proof of Theorem 3.3: When $n = 1$, clearly there are minimum monomial proper maps from Δ to $\mathbb{B}^N$ for any $N \geq 1$. When $n \geq 2$, by Theorem 3.7 there is always a minimum monomial proper map from $\mathbb{B}^n$ to $\mathbb{B}^{kn}$ for any $k \geq 1$. Then Theorem 3.3 is a consequence of Theorem 3.4.

4 Inequivalent families of holomorphic isometries

Let I_{n+k} be the maps defined in Section 3. In this section, we will make use of I_{n+k} to give inequivalent families of holomorphic isometries from $\mathbb{B}^n$ to $D_{n+k}^{IV}, 2 \leq k \leq n+2$. To emphasize the dependence on θ, we will write $I_{n+k,\theta}$ instead of I_{n+k}. More precisely, for $\theta \in (0, \pi/4]$, define

$$I_{n+2,\theta}(z) = \left(\cos\theta z_1, \sqrt{-1}\sin\theta z_1, z_2, \cdots, z_n, 1 - \sqrt{1 - \cos(2\theta)z_1^2 - \sum_{j=2}^{n} z_j^2} \right).$$

Fixing $\beta \in (0, \pi/4)$, for $\theta \in [\beta, \pi/4]$ define

$$I_{n+3,\theta}(z) = \left(\cos\theta z_1, \sqrt{-1}\sin\theta z_1, \cos\beta z_2, \sqrt{-1}\sin\beta z_2, z_3, \cdots, z_n, \right.$$

$$\left. 1 - \sqrt{1 - \cos(2\theta)z_1^2 - \cos 2\beta z_2^2 - \sum_{j=2}^{n} z_j^2} \right).$$

Similarly, we define $I_{n+k,\theta} : \mathbb{B}^n \to D_{n+k}^{IV}$ for all $3 \leq k \leq n+1$:

$$I_{n+k,\theta} = \left(\cos\theta z_1, \sqrt{-1}\sin\theta z_1, \cos\beta z_2, \sqrt{-1}\sin\beta z_2, \cdots, \cos\beta z_{k-1}, \sqrt{-1}\sin\beta z_{k-1}, \right.$$

$$\left. z_k, \cdots, z_n, 1 - \sqrt{1 - \cos(2\theta)z_1^2 - \cos(2\beta)\sum_{j=2}^{k-1} z_j^2 - \sum_{j=k}^{n} z_j^2} \right)$$

for $0 < \beta \leq \theta \leq \pi/4$. Here when $k = n+1$, the components "$z_k, \cdots, z_n$" is understood to be void. Fixing $\alpha \in (0, \pi/4)$ and $\beta \in (0, \pi/4)$, we define

$$I_{2n+2,\theta} = \left(\cos\theta z_1, \sqrt{-1}\sin\theta z_1, \cos\beta z_2, \sqrt{-1}\sin\beta z_2, \cdots, \cos\beta z_n, \sqrt{-1}\sin\beta z_n, \right.$$

$$\frac{\cos\alpha}{\cos(2\alpha)}\left(1 - \sqrt{1 - \cos(2\alpha)\left(\cos(2\theta)z_1^2 + \cos(2\beta)\sum_{j=2}^{n} z_j^2 \right)} \right),$$

$$\left. \frac{\sqrt{-1}\sin\alpha}{\cos(2\alpha)}\left(\sqrt{1 - \cos(2\alpha)\left(\cos(2\theta)z_1^2 + \cos(2\beta)\sum_{j=2}^{n} z_j^2 \right)} - 1 \right) \right)$$

for $\beta \leq \theta \leq \pi/4$. For $n \geq 2, 2 \leq k \leq n+2$, we will show that $I_{n+k,\theta}$ gives a real parameter family of inequivalent minimum holomorphic isometries from $\mathbb{B}^n$ to D_{n+k}^{IV}. More precisely,

Theorem 4.1. *Let $n \geq 2$. Then the following statements holds.*

- *$\{I_{n+2,\theta}\}_{0 < \theta \leq \pi/4}$ is a family of mutually inequivalent minimum holomorphic isometries.*

- *For each $2 < k \leq n+2$ and fixed $\beta \in (0, \pi/4)$, $\{I_{n+k,\theta}\}_{\beta \leq \theta \leq \pi/4}$ is a family of mutually inequivalent minimum holomorphic isometries.*

More precisely, for each $2 \leq k \leq n+2$, I_{n+k,θ_1} is equivalent to I_{n+k,θ_2} if and only if $\theta_1 = \theta_2$.

Proof. We will merely prove the case $k = 2$ and the remaining cases follow by similar argument. Let $0 < \theta_2 < \theta_1 \leq \pi/4$. Then we show that I_{n+2,θ_1} and I_{n+2,θ_2} are not equivalent.

Apply the Borel embedding to embed $\mathbb{B}^n$ as an open subset of $\mathbb{P}^n$ and D_{n+2}^{IV} as an open subset of $\mathbb{Q}^{n+2} \subset \mathbb{P}^{n+3}$ as before and write $[z, s] = [z_1, \cdots, z_n, s]$ to denote the homogeneous coordinates in $\mathbb{P}^n$. Under the homogeneous coordinates, $I_{n+2,\theta}$ can be identified with

$$\mathcal{I}_{n+2,\theta}(z, s) = [\phi_{1,\theta}(z, s), \cdots, \phi_{n+4,\theta}(z, s)]$$

from $\mathbb{P}^n$ to $\mathbb{P}^{n+3}$, where

$$\phi_{1,\theta}(z, s) = \cos\theta z_1;$$
$$\phi_{2,\theta}(z, s) = \sqrt{-1}\sin\theta z_1;$$
$$\phi_{j,\theta}(z, s) = z_{j-1} \text{ for } 3 \leq j \leq n+1;$$
$$\phi_{n+2,\theta}(z, s) = s - \sqrt{H_\theta(z, s)};$$
$$\phi_{n+3,\theta}(z, s) = \frac{1}{\sqrt{2}}\left(2s - \sqrt{H_\theta(z, s)}\right);$$
$$\phi_{n+4,\theta}(z, s) = \frac{1}{\sqrt{-2}}\sqrt{H_\theta(z, s)},$$

where $H_\theta(z, s) = s^2 - \cos(2\theta)z_1^2 - \sum_{j=2}^n z_j^2$. Note that for any $\theta \in [0, \pi/4)$ and $n \geq 2$, H_θ is irreducible and in particular, H_θ is not a perfect square of a polynomial in (z, s).

By the previous argument, I_{n+2,θ_1} is equivalent to I_{n+2,θ_2} if and only if there exist $\mathbf{U} \in U(n, 1)$ and $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Aut}(D_{n+2}^{IV})$ such that

$$(\phi_{1,\theta_1}, \cdots, \phi_{n+2,\theta_1})((z, s)\mathbf{U}) \equiv \frac{(\phi_{1,\theta_2}, \cdots, \phi_{n+2,\theta_2})A + (\phi_{n+3,\theta_2}, \phi_{n+4,\theta_2})C}{((\phi_{1,\theta_2}, \cdots, \phi_{n+2,\theta_2})B + (\phi_{n+3,\theta_2}, \phi_{n+4,\theta_2})D)(1/\sqrt{2}, \sqrt{-1/2})^t}(z, s). \tag{6}$$

Consequently, one has

$$\sqrt{H_{\theta_1}((z, s)\mathbf{U})} = R_1(z, s)\sqrt{H_{\theta_2}(z, s)} + R_2(z, s)$$

for rational functions $R_1(z, s), R_2(z, s)$. This is impossible by algebra if the following claim is true.

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Claim: For any $\mathbf{U} \in U(n, 1)$, $H_{\theta_1}((z, s)\mathbf{U})$ and $H_{\theta_2}(z, s)$ are coprime.

Proof of Claim: Suppose not. Since they are both irreducible, then there exists $\mathbf{U} \in U(n, 1)$ such that

$$H_{\theta_1}((z, s)\mathbf{U}) = cH_{\theta_2}(z, s) \tag{7}$$

for some nonzero complex number c. Write $H_\theta = -(z, s)A_\theta(z, s)^t$ with $A_\theta = \mathrm{diag}(\cos(2\theta), 1, \cdots, 1, -1)$. Then (7) yields that

$$\mathbf{U} \cdot A_{\theta_1} \cdot \mathbf{U}^t = c \cdot A_{\theta_2}.$$

This is impossible by Proposition 4.2. This finishes the proof of the claim. $\qquad\square$

Proposition 4.2. *Let $\lambda, \lambda_1, \cdots, \lambda_{n+1}$ be real numbers such that $|\lambda| < |\lambda_1| \le \cdots \le |\lambda_{n+1}|$ for $n \ge 1$. Then there does not exist an $(n+1) \times (n+1)$ matrix $\mathbf{U} \in U(n, 1)$, such that*

$$\mathbf{U} \cdot \mathrm{diag}(\lambda, \lambda_2, \cdots, \lambda_{n+1}) \cdot \mathbf{U}^t = c \cdot \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_{n+1}) \tag{8}$$

for some complex number c.

Proof. Write

$$\mathbf{U} = \begin{bmatrix} a_1 & b_1 & c_1 & \cdots \\ a_2 & b_2 & c_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+1} & b_{n+1} & c_{n+1} & \cdots \end{bmatrix}$$

and note that $\{a_1, a_2, \cdots, a_{n+1}\}$ cannot be all zero.

Claim: Only one element in $\{a_1, \cdots, a_{n+1}\}$ is not zero.

Proof of Claim: We will merely present the proof for $n = 3$ and the general case is similar. Suppose that the claim is not true. Then any vector (a_i, a_j, a_k) for $1 \le i < j < k \le 4$ is a nonzero vector. We now claim

$$a_4 \cdot \det \overline{\begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix}} = -b_4 \cdot \det \overline{\begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix}}, \tag{9}$$

$$a_4 \cdot \det \overline{\begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix}} = c_4 \cdot \det \overline{\begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix}} \tag{10}$$

$$a_4 \cdot \det \overline{\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}} = d_4 \cdot \det \overline{\begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix}} \tag{11}$$

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We only prove (9) and two others are similar. Note if both $\det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix}$ and $\det \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix}$ are zero, then (9) holds trivially. Without loss of generality, assume $\det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix} \neq 0$. The case when $\det \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix} \neq 0$ can be proved similarly. It follows from $\mathbf{U} \in U(n,1)$ that

$$(a_4, b_4, c_4, -d_4) \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ \bar{c}_1 & \bar{c}_2 & \bar{c}_3 \\ \bar{d}_1 & \bar{d}_2 & \bar{d}_3 \end{bmatrix} = (0,0,0).$$

This implies that

$$\begin{bmatrix} \bar{a}_1 & \bar{b}_1 & \bar{c}_1 & \bar{d}_1 \\ \bar{a}_2 & \bar{b}_2 & \bar{c}_2 & \bar{d}_2 \\ \bar{a}_3 & \bar{b}_3 & \bar{c}_3 & \bar{d}_3 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_4 \\ b_4 \\ c_4 \\ -d_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_4 \end{bmatrix}.$$

Namely, $(a_4, b_4, c_4, -d_4)^t$ is the solution of the linear system:

$$\begin{bmatrix} \bar{a}_1 & \bar{b}_1 & \bar{c}_1 & \bar{d}_1 \\ \bar{a}_2 & \bar{b}_2 & \bar{c}_2 & \bar{d}_2 \\ \bar{a}_3 & \bar{b}_3 & \bar{c}_3 & \bar{d}_3 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_4 \end{bmatrix}.$$

By the Cramer's rule, we know

$$a_4 = -b_4 \cdot \det \begin{bmatrix} \bar{b}_1 & \bar{c}_1 & \bar{d}_1 \\ \bar{b}_2 & \bar{c}_2 & \bar{c}_2 \\ \bar{b}_3 & \bar{c}_3 & \bar{d}_3 \end{bmatrix} / \det \begin{bmatrix} \bar{a}_1 & \bar{c}_1 & \bar{d}_1 \\ \bar{a}_2 & \bar{c}_2 & \bar{c}_2 \\ \bar{a}_3 & \bar{c}_3 & \bar{d}_3 \end{bmatrix}.$$

This implies (9).

We further claim:

$$\frac{\lambda}{\lambda_2} a_4 \cdot \det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix} = -b_4 \cdot \det \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix}, \tag{12}$$

$$\frac{\lambda}{\lambda_3} a_4 \cdot \det \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix} = c_4 \cdot \det \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix} \tag{13}$$

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$$\frac{\lambda}{\lambda_4} a_4 \cdot \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = -d_4 \cdot \det \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix} \tag{14}$$

We only prove (12) and two others are similar. Note again if both $\det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix}$ and

$\det \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix}$ are zero, then (10) holds trivially. Without loss of generality, assume $\det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix} \neq$

0. The case $\det \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix} \neq 0$ can be proved similarly. It follows from (8) that

$$(\lambda a_4, \lambda_2 b_4, \lambda_3 c_4, \lambda_4 d_4) \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix} = (0, 0, 0).$$

This implies:
$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda a_4 \\ \lambda_2 b_4 \\ \lambda_3 c_4 \\ \lambda_4 d_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \lambda_2 b_4 \end{bmatrix}.$$

Hence, (12) follows from the Cramer's rule.

Equations (9) and (12) imply that $a_4 \cdot \det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix}$ and $\frac{\lambda}{\lambda_2} a_4 \cdot \det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix}$ have the

same norm. However, $|\lambda/\lambda_2| < 1$. If follows that

$$a_4 \cdot \det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix} = 0. \tag{15}$$

Similarly, (10), (13) imply

$$a_4 \cdot \det \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix} = 0 \tag{16}$$

and (11), (14) imply that

$$a_4 \cdot \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = 0. \tag{17}$$

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Note that $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$ has rank 3 and $(a_1, a_2, a_3)^t$ is not zero. Then $\det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix}$,

$\det \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix}$ and $\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ cannot be all zero. This together with (15)-(17) implies

that $a_4 = 0$. Similar argument will yield $a_1 = a_2 = a_3 = 0$. This is a contradiction and the claim is thus proved.

We now assume that $a_{j_0} \neq 0$ for some $1 \leq j_0 \leq n+1$ and all other $a_j = 0$. It follows from $\mathbf{U} \in U(n, 1)$ that $|a_{j_0}| = 1$. Write $a_{j_0} = e^{\sqrt{-1}\theta}$ for $\theta \in [0, 2\pi)$. Note $\mathbf{U} \in U(n, 1)$ implies $\overline{\mathbf{U}}^{\mathbf{t}} \in U(n, 1)$. Write $\mathbf{u}_i$ as the ith column of $\mathbf{U}$. $\overline{\mathbf{U}}^{\mathbf{t}} \in U(n, 1)$ implies

$$\overline{\mathbf{u}}_i \cdot \mathrm{diag}(1, \cdots, 1, -1) \cdot \mathbf{u}_j = 0, \text{ if } j \neq 1. \tag{18}$$

We conclude from (18) the j_0-th row of $\mathbf{U}$ is $(e^{\sqrt{-1}\theta}, 0, \cdots, 0)$. Interchange the first and j_0-th row of $\mathbf{U}$ and still denote the new matrix by $\mathbf{U}$. Hence one has

$$\mathbf{U} = \begin{bmatrix} e^{\sqrt{-1}\theta} & \mathbf{0}_{1 \times n} \\ \mathbf{0}_{1 \times n}^t & \mathbf{V} \end{bmatrix} \tag{19}$$

and

$$\mathbf{U} \cdot \mathrm{diag}(\lambda, \lambda_2, \cdots, \lambda_{n+1}) \cdot \mathbf{U}^t = c \cdot \mathrm{diag}(\lambda_{j_0}, \lambda_{j_1}, \cdots, \lambda_{j_n}), \tag{20}$$

where $\{j_1, \cdots, j_n\}$ is a permutation of $\{1, \cdots, n+1\} \setminus \{j_0\}$. It follows from (19), (20) that

$$e^{2\sqrt{-1}\theta}\lambda = c\lambda_{j_0} \tag{21}$$

and

$$\mathbf{V} \cdot \mathrm{diag}(\lambda_2, \cdots, \lambda_{n+1}) \cdot \mathbf{V}^t = c \cdot \mathrm{diag}(\lambda_{j_1}, \cdots, \lambda_{j_n}). \tag{22}$$

Recall that $|\lambda| < |\lambda_1| \leq \cdots \leq |\lambda_{n+1}|$. (21) implies that $|c| < 1$. Note $\det(\mathbf{V}) = 1$ as $\det(\mathbf{U}) = 1$. Therefore (22) implies

$$|c|^n = \prod_{j=2}^{n+1} |\lambda_j| / \prod_{k=1}^{n} |\lambda_{j_k}| \geq 1.$$

This is a contradiction and thus the proposition is proved. $\qquad\square$

Remark 4.3. *By a similar argument as in the the proof of Theorem 4.1, one can show that for any $2 \leq k \leq n+2$, $(I_{n+k,\theta}, \mathbf{0})$, where θ varies in the given interval, is a family of mutually inequivalent holomorphic isometries from $\mathbb{B}^n$ to $D_m^{IV}, m \geq n+k$.*

5 Degree estimates

In this section, we prove various degree estimate results for holomorphic isometric maps from $\mathbb{B}^n$ to D_m^{IV}. We first introduce the following definition.

Definition 5.1. *Let F be a rational map from $\mathbb{C}^n$ into $\mathbb{C}^m$. We write*

$$F = \frac{(P_1, \cdots, P_m)}{R}$$

where $P_j, j = 1, \cdots, m$, and R are holomorphic polynomials and F is reduced to the lowest order term. The degree of F, denoted by $\deg(F)$, is defined to be

$$\deg(F) := max\{\deg(P_1), \cdots, \deg(P_m), \deg(R)\}.$$

Theorem 5.2. *Assume $m \geq n+1 \geq 3$. Let $F : \mathbb{B}^n \to D_m^{IV}$ be a rational holomorphic isometric embedding satisfying $F(0) = 0$. Then $\deg(F) \leq 2$. More precisely, F is either a linear map or $\deg(F) = 2$.*

Proof. Write $F = (f_1, \cdots, f_m)$ and $h = \frac{\sum_{i=1}^m f_i^2}{2}$. It follows from the isometry assumption that

$$\sum_{j=1}^m |f_j(z)|^2 = \sum_{j=1}^n |z_j|^2 + |h(z)|^2. \tag{23}$$

By a lemma of D'Angelo [D2], there exists a unitary matrix $\mathbf{U} = (u_{ij}) \in M(m, m; \mathbb{C})$ such that

$$\left(\frac{1}{2} \sum_{j=1}^m f_j^2(z), z_1, \cdots, z_n, 0, \cdots, 0 \right) \cdot \mathbf{U} = (f_1(z), \cdots, f_m(z)). \tag{24}$$

Equation (24) reads

$$f_j(z) = u_{1j} h(z) + \sum_{i=1}^n u_{i+1,j} z_i \tag{25}$$

for all $1 \leq j \leq m$. Take the sum of square of the above equations for all j, we conclude that

$$2h = \left(\sum_{j=1}^m u_{1j}^2 \right) h^2 + 2h \sum_{j=1}^m \left(u_{1j} \sum_{i=1}^n u_{i+1,j} z_i \right) + \sum_{j=1}^m \left(\sum_{i=1}^n u_{i+1,j} z_i \right)^2$$

$$= \left(\sum_{j=1}^m u_{1j}^2 \right) h^2 + 2h \sum_{j=1}^m \left(u_{1j} \sum_{i=1}^n u_{i+1,j} z_i \right) + \sum_{i=1}^n \left(\sum_{j=1}^m u_{i+1,j}^2 \right) z_i^2 + 2 \sum_{i \neq i'} \left(\sum_{j=1}^m u_{i+1,j} u_{i'+1,j} \right) z_i z_i'. \tag{26}$$

This is a quadratic equation

$$ah^2(z) + (p(z) - 2)h(z) + q(z) = 0 \tag{27}$$

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where $a = \sum_{j=1}^{n+1} u_{1j}^2$ and $p(z)$ and $q(z)$ are, if not identically zero, homogeneous polynomials in z of degree 1 and 2 respectively. Note that h is rational. We now prove that h must be a rational function of degree 2 in z or identically 0.

When $a \neq 0$, we split into two cases: $q(z) \equiv 0$ and $q(z) \not\equiv 0$. If $q(z) \not\equiv 0$, then by the quadratic formula,

$$h(z) = \frac{2 - p(z) \pm \sqrt{(p(z) - 2)^2 - 2q(z)}}{2a}.$$

This is a contradiction as $(p(z) - 2)^2 - 4q(z)$ cannot be a perfect sqaure of a polynomial. If $q(z) \equiv 0$, then we conclude by (27) that $h(z) \equiv 0$ as $h(0) = 0$. When $a = 0$, then $h = \frac{q(z)}{2 - p(z)}$. It must be either identically zero or a rational function of degree 2 in z. Here note $p(z) - 2$ and $q(z)$ must be coprime if $q(z) \not\equiv 0$.

If $h \equiv 0$, then f_j are linear polynomials for all j by (25). Now assume $h \not\equiv 0$. This corresponds to $a = 0$ and $h(z) = \frac{q(z)}{2 - p(z)}$ for $q \not\equiv 0$. We will show that $\deg(F) = 2$. If $p \equiv 0$, it is trivially true by (25). If $p \not\equiv 0$, again by (25),

$$f_j(z) = \frac{-u_{1j}q(z) + p(z)\left(\sum_{i=1}^{n} u_{i+1,j}z_i\right) - 2\left(\sum_{i=1}^{n} u_{i+1,j}z_i\right)}{p(z) - 2},$$

denoted by $\frac{N_j(z)}{p(z) - 2}$ for $1 \leq j \leq m$. We claim that there exists at least one j_0 such that $\deg(N_{j_0}(z)) = 2$. This claim will imply $\deg(f_{j_0}(z)) = 2$ as q cannot be divided by $p - 2$. We now give a proof of the claim. Suppose $\deg(N_j(z)) = 1$ for all $1 \leq j \leq m$. This is equivalent to

$$u_{1j}q(z) = p(z)\left(\sum_{i=1}^{n} u_{i+1,j}z_i\right), \quad \text{for all } 1 \leq j \leq m. \tag{28}$$

Assume $n \geq 2$. Then the matrix $\begin{bmatrix} u_{21}, \cdots, u_{2m} \\ \cdots, \cdots, \cdots \\ u_{(n+1)1}, \cdots, u_{(n+1)m} \end{bmatrix}$ is of rank equal to $n \geq 2$. Therefore, there exists $1 \leq j_1 < j_2 \leq m$ such that $\begin{bmatrix} u_{2j_1} \\ \cdots \\ u_{(n+1)j_1} \end{bmatrix}$ and $\begin{bmatrix} u_{2j_2} \\ \cdots \\ u_{(n+1)j_2} \end{bmatrix}$ are linearly independent and thus

$$\sum_{i=1}^{n} u_{i+1,j_1}z_i \not\equiv 0, \sum_{i=1}^{n} u_{i+1,j_2}z_i \not\equiv 0, \left(\sum_{i=1}^{n} u_{i+1,j_1}z_i, \sum_{i=1}^{n} u_{i+1,j_2}z_i\right) = 1.$$

It follows from (28) that $u_{1j_1} \neq 0, u_{1j_2} \neq 0$ and moreover,

$$q(z) = \frac{p(z)\left(\sum_{i=1}^{n} u_{i+1,j_1}z_i\right)}{u_{1j_1}} = \frac{p(z)\left(\sum_{i=1}^{n} u_{i+1,j_2}z_i\right)}{u_{1j_2}}.$$

This contradicts to the linear independence of $\begin{bmatrix} u_{2j_1} \\ \cdots \\ u_{(n+1)j_1} \end{bmatrix}$ and $\begin{bmatrix} u_{2j_2} \\ \cdots \\ u_{(n+1)j_2} \end{bmatrix}$.

Therefore we proved that either $\deg(F) = 2$ or F is a homogeneous linear polynomial map when $n \geq 2$. $\qquad \square$

We have a more precise result for $m < 2n$.

Theorem 5.3. *Assume $3 \leq n + 1 \leq m < 2n$. Let $F : \mathbb{B}^n \to D_m^{IV}$ be a rational holomorphic isometric embedding satisfying $F(0) = 0$. Then $\deg(F) = 2$.*

Proof. By Theorem 5.2, we have $\deg(F) = 1$ or 2. Hence we just need to show $\deg(F)$ cannot be 1. We will prove by seeking a contradiction. Suppose $\deg(F) = 1$. By the argument in Theorem 5.2, each $f_i, 1 \leq i \leq m$, is a homogeneous linear polynomial in z. Then $\sum_{j=1}^m f_j^2(z)$ will be a homogeneous quadratic polynomial in z. By collecting terms of degree 4 on both sides of (23), we have $\sum_{j=1}^m f_j^2(z) = 0$. Equation (24) is then reduced to,

$$(z_1, \cdots, z_n, 0, \cdots, 0)\, \mathbf{V} = (f_1(z), \cdots, f_m(z)). \tag{29}$$

Here $\mathbf{V} = \begin{pmatrix} \mathbf{v_1} \\ \dots \\ \mathbf{v_m} \end{pmatrix}$ is an $m \times m$ unitary matrix, $\mathbf{v_i}$ is an $m-$dimensional row vector, $1 \leq i \leq m$.

We rewrite (29) as

$$(z_1, ..., z_n) \begin{pmatrix} \mathbf{v}_1 \\ \dots \\ \mathbf{v}_n \end{pmatrix} = (f_1, ..., f_m). \tag{30}$$

The fact that $\sum_{j=1}^m f_j^2(z) = 0$, implies

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0, \ \text{for all } 1 \leq i, j \leq n. \tag{31}$$

As $\mathbf{V}$ is an unitary matrix, we have,

$$\mathbf{v}_i \cdot \overline{\mathbf{v}}_j = 0, \ \text{for all } 1 \leq i \neq j \leq n; \tag{32}$$

$$\mathbf{v}_i \cdot \overline{\mathbf{v}}_i = 1, 1 \leq i \leq n. \tag{33}$$

It follows from equations (31), (32) and (33) that

$$\mathrm{Re}\mathbf{v}_i \cdot \mathrm{Im}\mathbf{v}_i = 0, \mathrm{Re}\mathbf{v}_i \cdot \mathrm{Re}\mathbf{v}_i = \mathrm{Im}\mathbf{v}_i \cdot \mathrm{Im}\mathbf{v}_i = \frac{1}{2}, 1 \leq i \leq n. \tag{34}$$

$$\mathrm{Re}\mathbf{v}_i \cdot \mathrm{Im}\mathbf{v}_j = 0 \ \text{for all } 1 \leq i, j \leq n. \tag{35}$$

We thus get a collection of $2n$ mutually orthogonal nonzero real vectors $\{\mathrm{Re}\mathbf{v}_i, \mathrm{Im}\mathbf{v}_i\}_{i=1}^n$ in $\mathbb{C}^m$. This contradicts the assumption that $m < 2n$, thus establishes Theorem 5.3. $\qquad \square$

Remark 5.4. *We remark that the assumption $m < 2n$ is optimal in Theorem 5.3. Indeed, when $m = 2n$, we have a linear map $F : \mathbb{B}^n \to D_{2n}^{IV}$:*

$$F(z) = (\frac{\sqrt{2}}{2}z_1, \frac{\sqrt{-2}}{2}z_1, ..., \frac{\sqrt{2}}{2}z_n, \frac{\sqrt{-2}}{2}z_n).$$

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Furthermore, we have the following rigidity result for holomorphic rational isometric map of degree one.

Proposition 5.5. *Assume $m \geq 2n$ and $n \geq 2$. Let $F : \mathbb{B}^n \to D_m^{IV}$ be a rational holomorphic isometric embedding satisfying $F(0) = 0$. Assume that $\deg(F) = 1$. Then F is a totally geodesic embedding that is isotropically equivalent to*

$$\left(\frac{\sqrt{2}}{2} z_1, \frac{\sqrt{-2}}{2} z_1, ..., \frac{\sqrt{2}}{2} z_n, \frac{\sqrt{-2}}{2} z_n, 0, ..., 0 \right). \tag{36}$$

Proof. Recall from the proof of Theorem 5.2 and 5.3, we have if $\deg(F) = 1$, then F is a homogeneous linear map. More precisely, there is an $m \times m$ unitary matrix $V = \begin{pmatrix} \mathbf{v}_1 \\ ... \\ \mathbf{v}_m \end{pmatrix}$ such that equations (29)-(35) hold.

We write $\mathbf{a}_i = \operatorname{Re}\mathbf{v}_i, \mathbf{b}_i = \operatorname{Im}\mathbf{v}_i, 1 \leq i \leq n$, and write the $2n \times m$ matrix,

$$\mathbf{C} = \sqrt{2} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{b}_1 \\ ... \\ ... \\ \mathbf{a}_n \\ \mathbf{b}_n \end{pmatrix}.$$

As a consequence of (34), (35), we have

$$\mathbf{C}\mathbf{C}^t = \mathbf{I}_n.$$

We extend $\{\sqrt{2}\mathbf{a}_j, \sqrt{2}\mathbf{b}_j\}_{j=1}^n$ to an orthonormal basis $\{\sqrt{2}\mathbf{a}_1, \sqrt{2}\mathbf{b}_1, ..., \sqrt{2}\mathbf{a}_n, \sqrt{2}\mathbf{b}_n, \mathbf{c}_{2n+1}, ..., \mathbf{c}_m\}$ of $\mathbb{R}^m$. We then set $\widetilde{\mathbf{C}}$ to be the $m \times m$ matrix,

$$\widetilde{\mathbf{C}}^t = \begin{pmatrix} \sqrt{2}\mathbf{a}_1 \\ \sqrt{2}\mathbf{b}_1 \\ ... \\ ... \\ \sqrt{2}\mathbf{a}_n \\ \sqrt{2}\mathbf{b}_n \\ \mathbf{c}_{2n+1} \\ ... \\ \mathbf{c}_m \end{pmatrix},$$

then

$$\widetilde{\mathbf{C}}\widetilde{\mathbf{C}}^t = \mathbf{I}_m. \tag{37}$$

That is $\widetilde{\mathbf{C}} \in O(m, \mathbb{C})$. We now define
$$\widetilde{F} = F\widetilde{\mathbf{C}}^t.$$

Then $\widetilde{F}$ is orthogonal equivalent to F. Moreover,

$$\widetilde{F} = (z_1, ..., z_n) \begin{pmatrix} \mathbf{v}_1 \\ ... \\ \mathbf{v}_n \end{pmatrix} \widetilde{\mathbf{C}}^t = (z_1, ..., z_n) \begin{pmatrix} \mathbf{a}_1 + \sqrt{-1}\mathbf{b}_1 \\ ... \\ \mathbf{a}_n + \sqrt{-1}\mathbf{b}_n \end{pmatrix} \widetilde{\mathbf{C}}^t. \tag{38}$$

It then follows from (37) that

$$\widetilde{F} = \left(\frac{\sqrt{2}}{2} z_1, \frac{\sqrt{-2}}{2} z_1, ..., \frac{\sqrt{2}}{2} z_n, \frac{\sqrt{-2}}{2} z_n, 0, ..., 0 \right).$$

$\square$

6 Holomorphic maps to classical domains

In this section we construct proper holomorphic maps from $\mathbb{B}^n$ to an irreducible classical domains Ω. If $n < n_\Omega$, our construction gives many examples of non-isometric proper maps. If $n = n_\Omega$, our examples become non-totally geodesic isometric maps. The existence of these non-totally geodesic holomorphic isometries was first discovered by Mok [M6].

6.1 Type I domains

Let $q \geq p$. We recall that the type I domain is defined by

$$D_{p,q}^I = \{Z \in M(p, q; \mathbb{C}) | I_p - Z\overline{Z}^t > 0\}$$

and the Bergman kernel is given by

$$K(Z, \bar{Z}) = c_I \left(\det(I_p - Z\overline{Z}^t) \right)^{-(p+q)},$$

for some constant c_I depending on p, q. The boundary of $D_{p,q}^I$ is contained in

$$\{Z \in M(p, q; \mathbb{C}) | \det(I_p - Z\overline{Z}^t) = 0\}.$$

We will need the following lemma in algebra(cf. [H2]). We will denote by $Z\binom{i_1 \ \cdots \ i_k}{j_1 \ \cdots \ j_k}$ the determinant of the submatrix of Z formed by its $i_1^{\text{th}}, ..., i_k^{\text{th}}$ rows and $j_1^{\text{th}}, ..., j_k^{\text{th}}$ columns.

Lemma 6.1. *Let I_p be the $p \times p$ identity matrix $(p \geq 1)$, Z be a matrix as above.*

$$\det(I_p - Z\overline{Z}^t) = 1 + \sum_{k=1}^{p}(-1)^k \left(\sum_{1 \leq i_1 < i_2 < ... < i_k \leq p, 1 \leq j_1 < j_2 < ... < j_k \leq q} \left| Z\binom{i_1 \ \cdots \ i_k}{j_1 \ \cdots \ j_k} \right|^2 \right). \tag{39}$$

Write z as the coordinates in $\mathbb{C}^n$. Let $G(z)$ be a proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^{p+q-1}, q \geq p \geq 2, p+q-1 \geq n$, with $G(0) = 0$. Write $G = (g_1, ..., g_q, h_2, ..., h_p)$. We define a map H_G from $\mathbb{B}^n$ to $M(p, q; \mathbb{C})$ associated to G as follows.

$$H_G(z) = \begin{pmatrix} g_1 & g_2 & \cdots & g_q \\ h_2 & f_{22} & \cdots & f_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ h_p & f_{p2} & \cdots & f_{pq} \end{pmatrix}, \tag{40}$$

where

$$f_{ij} = \frac{h_i g_j}{g_1 - 1}, 2 \leq i \leq p, 2 \leq j \leq q.$$

We first note that

$$\begin{pmatrix} f_{2j} \\ \cdots \\ f_{pj} \end{pmatrix} = \frac{g_j}{g_1 - 1} \begin{pmatrix} h_2 \\ \cdots \\ h_p \end{pmatrix}, 2 \leq j \leq q;$$

$$\left(f_{i2}, \cdots, f_{iq} \right) = \frac{h_i}{g_1 - 1} \left(g_2, \cdots, g_q \right), 2 \leq i \leq p.$$

Consequently, the only submatrices of H_G possibly with nonzero determinant other than single entries are

$$H_G \begin{pmatrix} 1 & k \\ 1 & l \end{pmatrix} = f_{kl}, 2 \leq k \leq p, 2 \leq l \leq q.$$

As before, we denote by $H_G \begin{pmatrix} 1 & k \\ 1 & l \end{pmatrix}$ the determinant of the submatrix of H_G formed by the $1^{\text{st}}, k^{\text{th}}$ rows and $1^{\text{st}}, l^{\text{th}}$ columns.

Then by Lemma 6.1, we have

$$\det(I_p - H_G \overline{H_G}^t) = 1 - \sum_{i=2}^{p} |h_i|^2 - \sum_{j=1}^{q} |g_j|^2. \tag{41}$$

We claim that H_G maps $\mathbb{B}^n$ to $D_{p,q}^I$. Indeed, note $H_G(0) = 0$, and $\det(I_p - H_G \overline{H_G}^t) > 0$ in $\mathbb{B}^n$ by equation (41). Thus the claim follows easily from the path-connectedness of $\mathbb{B}^n$. Then the following proposition is a consequence of equation (41).

Proposition 6.2. *Let $q \geq p \geq 2, p+q \geq n+1$. H_G defined above is a proper holomorphic map from $\mathbb{B}^n$ to $D_{p,q}^I$.*

When $p + q - 1 \geq n + 1$, there is a proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^{p+q-1}$ that does not have a C^2-smooth extension up to any open piece of $\partial \mathbb{B}^n$ (cf. [Do]). In particular, it is not isometries with respect to Bergman metrics . Let G be such a map. Then we have,

Theorem 6.3. *Let $p + q \geq n + 2$. Then there exists a proper holomorphic map from $\mathbb{B}^n$ to $D^I_{p,q}$ that does not extend C^2-smoothly up to any open piece of boundary $\partial \mathbb{B}^n$. In particular, it is not isometric.*

We next consider the case when $n = n_{D^I_{p,q}} = p + q - 1$. Write $z = (z_1, ..., z_q, w_2, ..., w_p)$ as the coordinates in $\mathbb{C}^{p+q-1}$. Let $G(z) = z$ be the identity map from $\mathbb{B}^{p+q-1}$ to $\mathbb{B}^{p+q-1}$.

Let $R^I_{p,q} = H_G$. Namely,

$$R^I_{p,q} = \begin{pmatrix} z_1 & z_2 & ... & z_q \\ w_2 & f_{22} & ... & f_{2q} \\ ... & ... & ... & ... \\ w_p & f_{p2} & ... & f_{pq} \end{pmatrix}, \tag{42}$$

where $f_{ij} = \frac{w_i z_j}{z_1 - 1}, 2 \leq i \leq p, 2 \leq j \leq q$. It is then straightforward to verify that $R^I_{p,q}$ is a holomorphic isometry from $\mathbb{B}^{p+q-1}$ to $D^I_{p,q}$. Indeed, by equation (41), we have,

$$\det(I_p - R^I_{p,q} \overline{R^I_{p,q}}^t) = 1 - \sum_{j=1}^{q} |z_i|^2 - \sum_{i=2}^{p} |w_i|^2.$$

We end this subsection with the following question:

Question 6.4. *Is every proper holomorphic map from $\mathbb{B}^{p+q-1}$ with $q \geq p \geq 2, p + q \geq 4$ to $D^I_{p,q}$ an isometry?*

Note when $p = 1$ in the above question, we have $D^I_{1,q} = \mathbb{B}^q$. Thus the answer to the question is positive if $p = 1$ by [Al].

6.2 Type II domains

The type II domain is defined by

$$D^{II}_m = \{Z \in D^I_{m,m} | Z = -Z^t\}$$

and the Bergman kernel is given by

$$K(z, \bar{z}) = c_{II} \left(\det(I_m - Z\overline{Z}^t) \right)^{-(m-1)},$$

for some positive constant c_{II} depending on m. Its boundary is contained in

$$\{Z \in M(m, m; \mathbb{C}) | \det(I_m - Z\overline{Z}^t) = 0\}.$$

We will need the following results from algebra.

Lemma 6.5. *Let $A = (a_{ij})$ be a $2n \times 2n, n \geq 1$, skew-symmetric matrix. Then*

$$\det(A) = (pf(A))^2.$$

Here $\mathrm{pf}(A)$ is a homogenous polynomial in the matrix entries. This polynomial is called the Pfaffian of the matrix A that can be explicitly written as follows. Let Π be the set of all partitions of $\{1, 2, ..., 2n\}$ into pairs without regard to order. An element $\alpha \in \Pi$ can be written as

$$\alpha = \{(i_1, j_1), (i_2, j_2), ..., (i_n, j_n)\}$$

with $i_k < j_k$ and $i_1 < i_2 < ... < i_n$. Let

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & ... & 2n \\ i_1 & j_1 & i_2 & j_2 & ... & j_n \end{bmatrix}$$

be the corresponding permutation. Given a partition α as above, define

$$A_\alpha = \mathrm{sgn}(\pi) a_{i_1, j_1} a_{i_2, j_2} \cdots a_{i_n, j_n}.$$

The Pfaffian of A is then given by,

$$\mathrm{pf}(A) = \sum_{\alpha \in \Pi} A_\alpha.$$

Note that the determinant of an $n \times n$ skew-symmetric matrix for n odd is always zero. The Pfaffian of an $n \times n$ skew-symmetric matrix for n odd is defined to be zero.

Moreover, we have the following result from algebra. For more details and its proof, see [H2], [PS].

Lemma 6.6. *Let I_n be the $n \times n$ identity matrix, Z be an $n \times n$ skew-symmetric matrix. Then*

$$\det(I_n - Z\overline{Z}^t) = \left(1 + \sum_{1 \leq k \leq n, 2|k} (-1)^{\frac{k}{2}} \left(\sum_{1 \leq i_1 < ... < i_k \leq n} \left| Z \begin{pmatrix} i_1 & ... & i_k \\ i_1 & ... & i_k \end{pmatrix} \right| \right) \right)^2. \tag{43}$$

Here "$2|k$" means that k is divisible by 2.

Write z as the coordinates in $\mathbb{C}^n$. Let $G(z)$ be a proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^{2m-3}$, where m is an integer with $2m - 3 \geq n, m \geq 3$. Assume $G(0) = 0$. Write $G = (g_2, ..., g_m, h_3, ..., h_m)$. We define a holomorphic map H_G from $\mathbb{B}^n$ to $M(m, m; \mathbb{C})$ associated to G by

$$H_G(z) = \begin{pmatrix} 0 & g_2 & g_3 & g_4 & ... & g_m \\ -g_2 & 0 & h_3 & h_4 & ... & h_m \\ -g_3 & -h_3 & 0 & f_{34} & ... & f_{3m} \\ ... & ... & ... & ... & ... & ... \\ -g_{m-1} & ... & ... & ... & 0 & f_{(m-1)m} \\ -g_m & -h_m & ... & ... & ... & 0 \end{pmatrix}, \tag{44}$$

where
$$f_{ij} = \frac{g_i h_j - g_j h_i}{g_2 - 1}, 3 \le i, j \le m.$$

Proposition 6.7. *Let $m \ge 3, 2m - 3 \ge n$. H_G in (44) is a proper holomorphic map from $\mathbb{B}^n$ to D_m^{II}.*

Proof. We first compute the determinants of the principal submatrices of H_G. It follows from the straightforward calculation that

$$H_G \begin{pmatrix} 1 & i \\ 1 & i \end{pmatrix} = (g_i)^2, 2 \le i \le m;$$

$$H_G \begin{pmatrix} 2 & j \\ 2 & j \end{pmatrix} = (h_j)^2, 3 \le j \le m;$$

$$H_G \begin{pmatrix} i & j \\ i & j \end{pmatrix} = (f_{ij})^2, 3 \le i < j \le m.$$

Moreover,

$$H_G \begin{pmatrix} 1 & 2 & k & l \\ 1 & 2 & k & l \end{pmatrix} = (g_2 f_{kl} - g_k h_l + g_l h_k)^2 = (f_{kl})^2, \quad 3 \le k < l \le m; \tag{45}$$

$$H_G \begin{pmatrix} 1 & j & k & l \\ 1 & j & k & l \end{pmatrix} = (g_j f_{kl} - g_k f_{jl} + g_l f_{jk})^2 = 0, \quad 3 \le j < k < l \le m; \tag{46}$$

$$H_G \begin{pmatrix} 2 & j & k & l \\ 2 & j & k & l \end{pmatrix} = (h_j f_{kl} - h_k f_{jl} + h_l f_{jk})^2 = 0, \quad 3 \le j < k < l \le m; \tag{47}$$

$$H_G \begin{pmatrix} i & j & k & l \\ i & j & k & l \end{pmatrix} = (f_{ij} f_{kl} - f_{ik} f_{jl} + f_{jk} f_{il})^2 = 0, \quad 3 \le i < j < k < l \le m. \tag{48}$$

For higher order submatrices, we have the lemma below.

Lemma 6.8. *Every principal submatrix of H_G with order ≥ 5 has zero determinant.*

Proof. First we note that any $m \times m$ anti-symmetric matrix for odd m has zero determinant. For m even, we recall the fact that the Pfaffian of an anti-symmetric $m \times m$ matrix A can be computed recursively as

$$\mathrm{pf}(A) = \sum_{j=2}^{m} (-1)^j a_{1j} \mathrm{pf}(A_{\hat{1}\hat{j}}).$$

Here $A_{\hat{1}\hat{j}}$ denotes the matrix obtained from A with both its 1-st and j-th rows and columns removed. This together with (47)-(48) yields that all principal submatrices of H_G with even order ≥ 6 has zero determinant. $\square$

Then it follows from Lemma 6.6 that

$$\det(I_n - H_G \overline{H_G}^t) = \left(1 - \sum_{i=2}^m |g_i|^2 - \sum_{j=3}^m |h_j|^2 \right)^2. \tag{49}$$

Therefore we conclude as in type I case that H_G is a proper holomorphic map from $\mathbb{B}^n$ to D_n^{II}. $\qquad\square$

When $2m - 3 \geq n+1$, i.e., $m \geq 2 + \frac{n}{2}$, there is a proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^{2m-3}$ that does not have a C^2-smooth extension to any open piece of $\partial \mathbb{B}^n$ (See [Do]). In particular, it is not isometric. Let G be such a map. Then H_G is not isometric, either. We thus have proved,

Theorem 6.9. *Let m, n be integers such that $n \geq 2, m \geq 2 + \frac{n}{2}$. Then there is a proper holomorphic map from $\mathbb{B}^n$ to D_m^{II} that does not extend C^2-smoothly to any open piece of $\partial \mathbb{B}^n$. In particular, it is not isometric.*

Now we consider the case $n = n_{D_m^{II}} = 2m - 3$ where m is an integer. Write $z = (z_2, ..., z_m, w_3, ..., w_m)$ as the coordinates of $\mathbb{C}^{2m-3}, n \geq 5$. Let $G(z) = z$ be th identity map from $\mathbb{B}^{2m-3}$ to itself. In this case, H_G is reduced to $R_m^{II} : \mathbb{B}^{2m-3} \to D_m^{II}$ given by

$$R_m^{II} = \begin{pmatrix} 0 & z_2 & z_3 & z_4 & \ldots & z_m \\ -z_2 & 0 & w_3 & w_4 & \ldots & w_m \\ -z_3 & -w_3 & 0 & f_{34} & \ldots & f_{3m} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -z_{m-1} & \ldots & \ldots & \ldots & 0 & f_{(m-1)m} \\ -z_m & -w_m & \ldots & \ldots & \ldots & 0 \end{pmatrix}, \tag{50}$$

where $f_{ij} = \frac{z_i w_j - z_j w_i}{z_2 - 1}, 3 \leq i, j \leq m$. By equation (49), we have

$$\det(I_n - R_m^{II} \overline{R_m^{II}}^t) = (1 - \sum_{i=2}^m |z_i|^2 - \sum_{j=3}^m |w_j|^2)^2.$$

We thus conclude that R_m^{II} is a holomorphic isometry from $\mathbb{B}^{2m-3}$ to D_m^{II}.
We again ask the similar question:

Question 6.10. *Is every proper holomorphic map from $\mathbb{B}^{2m-3}, m \geq 4$, to D_m^{II} an isometry?*

Note when $m = 2$, the answer to the above question is negative as D_2^{II} is biholomorphically isometric to the unit disc. When $m = 3$, the answer to the question is positive as D_3^{II} is biholomorphically isometric to $\mathbb{B}^3$.

6.3 Type III domains

The type III dmain is defined by

$$D_m^{III} = \{Z \in D_{m,m}^I | Z = Z^t\}$$

and the Bergman kernel is given by

$$K(Z, \bar{Z}) = c_{III} \left(\det(I_m - Z\overline{Z}^t) \right)^{-(m+1)}$$

for some constant c_{III} depending on m. Its boundary is contained in

$$\{Z \in M(m, m; \mathbb{C}) | \det(I_m - Z\overline{Z}^t) = 0\}.$$

We write $z = (z_1, ..., z_n)$ as the coordinates in $\mathbb{C}^n$. Let $G(z)$ be a proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^m$, $m \geq \max\{n, 2\}$, with $G(0) = 0$. Write $G = (g_1, ..., g_m)$. We define a holomorphic map H_G from $\mathbb{B}^n$ to $M(m, m; \mathbb{C})$ associated to G :

$$H_G(z) = \begin{pmatrix} g_1 & \frac{g_2}{\sqrt{2}} & \cdots & \frac{g_m}{\sqrt{2}} \\ \frac{g_2}{\sqrt{2}} & f_{22} & \cdots & f_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{g_m}{\sqrt{2}} & f_{m2} & \cdots & f_{mm} \end{pmatrix}, \tag{51}$$

where

$$f_{ij} = f_{ji} = \frac{g_i g_j}{2(g_1 - 1)}, 2 \leq i, j \leq m.$$

Then we can prove the following proposition similarly.

Proposition 6.11. *Let $m \geq \max\{n, 2\}$, H_G be as above. Then H_G is a proper holomorphic proper map from $\mathbb{B}^n$ to D_m^{III}.*

Proof. We first compute determinants of all 2×2 submatrices of H_G.

$$H_G \begin{pmatrix} 1 & i \\ 1 & k \end{pmatrix} = f_{ik}, 2 \leq i, k \leq m;$$

$$H_G \begin{pmatrix} 1 & i \\ k & l \end{pmatrix} = H_G \begin{pmatrix} k & l \\ 1 & i \end{pmatrix} = 0, \ 2 \leq i \leq m, \ 2 \leq k < l \leq m;$$

$$H_G \begin{pmatrix} i & j \\ k & l \end{pmatrix} = 0, \ 2 \leq i < j \leq m, \ 2 \leq k < l \leq m.$$

This implies that the determinants of all 3×3 submatrices of H_G are zero. We then obtain by Lemma 6.1 that

$$\det(I_n - H_G \overline{H_G}^t) = 1 - \sum_{i=1}^{m} |g_i|^2. \tag{52}$$

Also note $H_G(0) = 0$. We conclude as before that H_G is a proper holomorphic map from $\mathbb{B}^n$ to D_m^{III}. $\qquad \square$

Again since when $m \geq n + 1$, there is a proper holomorphic map G that does not have a C^2-smooth extension up to any open piece of $\partial\mathbb{B}^n$ (See [Do]). In particular, it is not isometric. Then H_G is not an isometry, either. Thus we have

Theorem 6.12. *Let $m \geq n+1$. Then there is a proper holomorphic map from $\mathbb{B}^n$ to D_m^{III} that does not extend C^2-smoothly to any open piece of $\partial\mathbb{B}^n$. In particular, it is not isometric.*

Next when $n = n_{D_m^{III}} = m$, we write $G(z) = z$ be the identity map from $\mathbb{B}^m$ to $\mathbb{B}^m$. Let $R_m^{III} = H_G$. Namely,

$$
R_n^{III} = \begin{pmatrix} z_1 & \frac{z_2}{\sqrt{2}} & \cdots & \frac{z_m}{\sqrt{2}} \\ \frac{z_2}{\sqrt{2}} & f_{22} & \cdots & f_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{z_m}{\sqrt{2}} & f_{m2} & \cdots & f_{mm} \end{pmatrix},
\tag{53}
$$

where $f_{ij} = f_{ji} = \frac{z_i z_j}{2(z_1-1)}, 2 \leq i, j \leq m$. It is easy to verify that R_m^{III} is a holomorphic isometry from $\mathbb{B}^m$ to D_m^{III} by (52).

We also raise the similar question:

Question 6.13. *Is every proper holomorphic map from $\mathbb{B}^m, m \geq 2$, to D_m^{III} an isometry?*

6.4 Type IV domains

The Type IV case was studied in [XY]. Nevertheless we record them here for completeness. The type IV domain is defined by

$$
D_m^{IV} = \{Z = (z_1, \cdots, z_m) \in \mathbb{C}^m | Z\overline{Z}^t < 2 \text{ and } 1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2 > 0\}
$$

and the Bergman kernel is given by

$$
K(Z, \bar{Z}) = c_{IV}\left(1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2\right)^{-m}
$$

for some positive constant c_{IV} depending on m. Its boundary is given by

$$
\{Z = (z_1, \cdots, z_m) \in \mathbb{C}^m | Z\overline{Z}^t \leq 2 \text{ and } 1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2 = 0\}.
$$

Write z as the coordinates in $\mathbb{C}^n$. Let G be a proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^m, m \geq n$, with $G(0) = 0$. Write $G = (g_1, ..., g_m)$. We define two holomorphic maps H_G, W_G from $\mathbb{B}^n$ to $\mathbb{C}^{m+1}$ associated to G :

$$
H_G = (f_1, ..., f_{m-1}, f_m, f_{m+1}),
\tag{54}
$$

29

where $f_i = g_i, 1 \leq i \leq m-1, f_m = \frac{P_m}{Q}, f_{m+1} = \frac{P_{m+1}}{Q}$,

$$P_m = \frac{1}{2}\sum_{i=1}^{m-1} g_i^2 - g_m^2 + g_m, P_{m+1} = \sqrt{-1}\left(\frac{1}{2}\sum_{i=1}^{m-1} g_i^2 + g_m^2 - g_m\right), Q = \sqrt{2}(1 - g_m),$$

and

$$W_G = \left(g_1, \cdots, g_{m-1}, g_m, 1 - \sqrt{1 - \sum_{j=1}^{m} g_j^2}\right). \tag{55}$$

By direct computation we have

$$1 - H_G\overline{H_G}^t + \frac{1}{4}|H_G H_G^t|^2 = 1 - W_G\overline{W_G}^t + \frac{1}{4}|W_G W_G^t|^2 = 1 - \sum_{i=1}^{m}|g_i|^2. \tag{56}$$

Hence as before we conclude

Proposition 6.14. *Let $m \geq n$, H_G, W_G be as above. Then H_G, W_G are both proper holomorphic maps from $\mathbb{B}^n$ to D_{m+1}^{IV}.*

Again if $m \geq n+1$, we can choose G to be a proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^m$ that does not have a C^2-smooth extension to any open piece of $\partial\mathbb{B}^n$(See [Do]). Then we have,

Theorem 6.15. *Let $m \geq n+1$. Then there is a proper holomorphic map from $\mathbb{B}^n$ to D_{m+1}^{IV} that does not extend C^2-smoothly to any open piece of $\partial\mathbb{B}^n$. In particular, it is not isometric.*

Now when $n = n_{D_{m+1}^{IV}} = m$, write $z = (z_1, ..., z_m)$ be the coordinates in $\mathbb{C}^m$ for $m \geq 2$. Let $G(z) = z$ be the identity map from $\mathbb{B}^m$ to $\mathbb{B}^m$. Then H_G, W_G are reduced to $R_m^{IV}, I_m^{IV} : \mathbb{B}^m \to D_{m+1}^{IV}$. Here

$$R_m^{IV} = (r_1, ..., r_{m-1}, r_m, r_{m+1}), \tag{57}$$

where $r_i = z_i, 1 \leq i \leq m-1, r_m = \frac{p_m}{q}, r_{m+1} = \frac{p_{m+1}}{q}$,

$$p_m = \frac{1}{2}\sum_{i=1}^{m-1} z_i^2 - z_m^2 + z_m, \quad p_{m+1} = \sqrt{-1}\left(\frac{1}{2}\sum_{i=1}^{m-1} z_i^2 + z_m^2 - z_m\right), \quad q = \sqrt{2}(1 - z_m).$$

$$I_m^{IV} = \left(z_1, \cdots, z_{m-1}, z_m, 1 - \sqrt{1 - \sum_{j=1}^{m} z_j^2}\right). \tag{58}$$

It is easy to see from equation (56) that R_m^{IV}, I_m^{IV} are both holomorphic isometries. We showed in [XY] that R_m^{IV}, I_m^{IV} are the only holomorphic isometries from $\mathbb{B}^m$ to D_{m+1}^{IV} up to holomorphic automorphisms. We also proved in [XY] that they are the only two proper holomorphic maps from $\mathbb{B}^m$ to D_{m+1}^{IV} satisfying certain boundary regularity when $m \geq 4$. In addition, the following question is posed in [XY]:

Question 6.16. *Is every proper holomorphic maps from $\mathbb{B}^m, m \geq 2$, to D_{m+1}^{IV} an isometry?*

As mentioned above, we gave a positive answer to this question when $m \geq 4$ in [XY] with an additional boundary regularity assumption. Also note the answer to the question is negative if $m = 1$ (See [XY]).

6.5 Singularities of holomorphic isometries

The rational holomorphic isometries given in previous sections from the unit ball $\mathbb{B}^{n_\Omega}$ into an irreducible classic symmetric domain Ω are not totally geodesic and only produce singularities at one single point on the boundary $\partial \mathbb{B}^{n_\Omega}$. When $n_\Omega \geq 2$, one can easily avoid passing through this point by slicing $\mathbb{B}^{n_\Omega}$ with a complex hyperplane. Therefore, one obtains holomorphic polynomial isometries from $\mathbb{B}^{n_\Omega - 1}$ into Ω. In particular, this answers the question raised by Mok in [M4] (Question 5.2.2) in the negative while the positive answer may still be possible from $\mathbb{B}^{n_\Omega}$ to Ω. Note that these examples are discovered independently by Chan-Mok [CM].

Theorem 6.17. *There exist non-totally geodesic holomorphic isometries from the unit ball $\mathbb{B}^m$ to the four types of irreducible bounded symmetric domain that extends holomorphically to $\mathbb{C}^m$.*

Proof. Holomorphic isometries F are given by the following maps for different targets.

$$
F(z_2, \cdots, z_q, w_2, \cdots, w_p) = \begin{pmatrix} 0 & z_2 & \ldots & z_q \\ w_2 & -w_2 z_2 & \ldots & -w_2 z_q \\ \ldots & \ldots & \ldots & \ldots \\ w_p & -w_p z_2 & \ldots & -w_p z_q \end{pmatrix} : \mathbb{B}^{p+q-2} \to D_{p,q}^I \text{ for } q \geq p \geq 2;
$$

$$
F(z_3, \cdots, z_n, w_3, \cdots, w_n) =
$$
$$
\begin{pmatrix} 0 & 0 & z_3 & z_4 & \ldots & z_n \\ 0 & 0 & w_3 & w_4 & \ldots & w_n \\ -z_3 & -w_3 & 0 & w_3 z_4 - z_3 w_4 & \ldots & z_n w_3 - z_3 w_n \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -z_{n-1} & \ldots & \ldots & \ldots & 0 & z_n w_{n-1} - z_{n-1} w_n \\ -z_n & -w_n & \ldots & \ldots & \ldots & 0 \end{pmatrix} : \mathbb{B}^{2n-4} \to D_n^{II} \text{ for } n \geq 4;
$$

$$
F(z_2, \cdots, z_n) = \begin{pmatrix} 0 & \frac{z_2}{\sqrt{2}} & \ldots & \frac{z_n}{\sqrt{2}} \\ \frac{z_2}{\sqrt{2}} & -\frac{z_2^2}{2} & \ldots & -\frac{z_2 z_n}{2} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{z_n}{\sqrt{2}} & -\frac{z_2 z_n}{2} & \ldots & -\frac{z_n^2}{2} \end{pmatrix} : \mathbb{B}^{n-1} \to D_n^{III} \text{ for } n \geq 2;
$$

$$
F(z_1, \cdots, z_{n-1}) = \left(z_1, \cdots, z_{n-1}, -\frac{\sqrt{2}}{4} \sum_{i=1}^{n-1} z_i^2, \frac{\sqrt{-2}}{4} \sum_{i=1}^{n-1} z_i^2 \right) : \mathbb{B}^{n-1} \to D_{n+1}^{IV} \text{ for } n \geq 2.
$$

Indeed, these are polynomial holomorphic isometries. $\square$

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