REGULARITY OF MAPPINGS INTO CLASSICAL DOMAINS

MING XIAO

Abstract. We study the regularity and algebraicity for mappings into classical domains. Among other things, we establish everywhere regularity and algebraicity results for C^2 CR maps into the smooth boundary of a classical domain where the codimension can be arbitrarily large.

2010 Mathematics Subject Classification: Primary 32H40 Secondary 32M15 32H35

1. Introduction

The first part of the article is devoted to establishing reflection principle type results for holomorphic and CR maps in several complex variables. The typical question of the reflection principle asks to find conditions under which a CR map between real analytic CR submanifolds in complex spaces extends holomorphically to an open neighborhood of the source manifold and when the submanifolds are merely smooth, we investigate when the map has C^∞ regularity. Results of this type date back to the work of Fefferman [Fe], Lewy [Le], and Pinchuk [Pi]. In this paper, we concentrate on CR mappings between real hypersurfaces of different dimensions. Much attention has been paid to the development of the reflection principle along this line since the pioneering work of Webster [W], Faran [Fa], and Forstnerič [Fr1]. We cannot mention all related work but only name a few recent work here: [La1-3], [BX1-2], [KLX], [Mi], [LM]. See the book by Baouendi-Ebenfelt-Rothschild [BER2] for a detailed account and more earlier references on this subject.

A particular case of interest is to study reflection principle type problem for maps between real algebraic CR manifolds. A CR manifold is called real algebraic if it is defined by real polynomials. The class of real algebraic CR manifolds is of fundamental importance in several complex analysis. They arises naturally as the boundaries of bounded symmetric domains and the tube domain of future light cone, as well as some homogeneous CR manifolds. Algebraicity properties of biholomorphisms between real algebraic CR submanifolds has been extensively studied (cf. [BER1-2] and references therein). Starting from the work of Forstnerič [Fr1] and Huang [Hu1], one expects a finitely smooth CR map between real algebraic CR manifolds of different dimensions also to be algebraic under some geometric conditions. Here a holomorphic map $F : U \subset \mathbb{C}^n \to \mathbb{C}^m$ is called (Nash) algebraic if each component of F satisfies some holomorphic polynomial. Huang ([Hu1]) proved a CR map $F : M \subset \mathbb{C}^n \to M' \subset \mathbb{C}^N (N > n > 1)$ of class C^{N-n+1} between strongly pseudoconvex real algebraic hypersurfaces M and M' must extend to an algebraic holomorphic map. In particular, if M and M' are spheres, then by the result of Forstnerič [Fr1], F must extend to a holomorphic rational map. Zaitsev [Za] established algebraicity results for holomorphic maps between quite general CR submanifolds. See the references in [Za] for more related algebraicity results.

The study of reflection principle is closely related to the regularity problem of proper maps between two domains of possibly different dimensions. The discovery of inner functions reveals that at least some initial regularity must be assumed to expect a reflection principle type result when the source domain is strongly pseudoconvex (cf. [HS], [Gl], [Fr2], [St]). An important and long-standing question in lots of settings is to find the optimal initial regularity assumption under which reflection principle holds. Let $F : \Omega_1 \subset \mathbb{C}^n \to \Omega_2 \subset \mathbb{C}^N$ be a holomorphic proper map between two domains. Even the following question has puzzled researchers since a long time ago: Does there exist an integer k which is independent of the codimension such that the reflection principle holds for proper maps F with C^k boundary regularity under certain geometric conditions? Results are known only in some special cases. Huang ([Hu3]) proved a proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^N$ with $1 < n < N \leq \frac{n(n+1)}{2}$, that is C^3-smooth up to a boundary point, must be rational. We obtain in this paper a similar algebraicity result for mappings into bounded symmetric domains. More precisely, we show the algebraicity of proper maps into certain type I classical domains where the boundary regularity is always assumed to be C^2 and the codimension can be arbitrary large.

The subject of bounded symmetric domains plays an important role in complex analysis and geometry. Many striking rigidity phenomena have been discovered on them or their quotient spaces. See the work of Siu [S1-2] and Mok [M1-2] as well as the book [M3] for a detailed account on this subject. The study of rigidity of proper maps between bounded symmetric domains dates back to the work of Poincaré [Po] and Alexander [Al] on proper self-maps of unit balls of dimension at least two. Many researchers have contributed to the study of proper maps between unit balls of different dimensions. To name a few, we mention here [W], [Fa], [Fr], [Hu2-3], [HJ], [E2], [DL], [DX1-2]. The study of maps between bounded symmetric domains of higher rank has a vast difference due to their distinct geometric structures. Tsai [Ts] proved that a proper map between bounded symmetric domains D_1 and D_2 of the same rank (at least rank two) must be a totally geodesically isometric embedding if D_1 is irreducible. Much less is known about proper maps between bounded symmetric domains of different ranks. Many interesting results have been obtained in the type I classical domain case, see Mok [M4], Tu [Tu1-2], Ng [Ng1-2], Kim-Zaitsev [KZ1-2], etc. We refer the readers to a recent article [NTY] for detailed surveys on this subject. Recently, Mok [M5] initiated the study of rigidity of maps from the unit ball to bounded symmetric domains of higher rank, followed by the work of [CM], [Ch], [UWZ], [XY1-2]. Our regularity results might shed light on future work towards understanding rigidity of proper maps into classical domains.

We first introduce the following reflection principle type result, which is a starting point of this paper. It recovers Theorem 1 in [KXL] as a special case. Recall a real hypersurface is called real-algebraic if it is defined by a real polynomial. A map $F\colon M \to M'$ between real hypersurfaces is said to be *CR-transversal* at $p \in M$ if $T^{(1,0)}_{F(p)}M' + T^{(0,1)}_{F(p)}M' + dF(\mathbb{C}T_pM) = \mathbb{C}T_{F(p)}M'$. The definition of uniform $2-$nondegeneracy will be given in Section 2.

Theorem 1. *Let $M \subset \mathbb{C}^{n+1}, (n \geq 1)$ be a strongly pseudoconvex real-analytic (resp. smooth) hypersurface, and $M' \subset \mathbb{C}^{N+1}, N > n$, a uniformly $2-$nondegenerate real-analytic (resp. smooth) hypersurface. Assume the Levi form of M' has exactly n nonzero eigenvalues at every point. Let $F : M \mapsto M'$ be a CR-transversal CR-mapping of class C^2. Then*

(1) F is real-analytic (resp. smooth) everywhere on M.
(2) If in addition M, M' are both real-algebraic, then F extends to an algebraic holomorphic map.

Remark 1.1. If there is a C^2 CR transversal map from a strongly pseudoconvex hypersurface $M \subset \mathbb{C}^{n+1}$ near $p \in M$ to $M' \subset \mathbb{C}^{N+1}$, then the Levi form of M' has at least n nonzero eigenvalues of the same sign near $F(p)$. Example 2.4 in Section 2 shows that if we allow the Levi form of M' to have more than n nonzero eigenvalues (even of the same sign) in Theorem 1, then the conclusions fail, even if one poses stronger initial smoothness assumption on F.

The main purpose of this paper is to find applications of Theorem 1 and establish regularity results for maps into Cartan's classical domains (or their boundaries). For that we systematically study the CR geometric properties of the smooth (part of the) boundaries of classical domains. Here for a semi-analytic set $A \subset \mathbb{C}^N$, we say $a \in A$ is a smooth point if there is a neighborhood U of a in $\mathbb{C}^N$ such that $A \cap U$ is a real-analytic submanifold in U. Otherwise, a is called a singular point. To explain our results, we first recall the definition of Hermitian symmetric spaces. A complex manifold X with a Hermitian metric h is said to be a Hermitian symmetric space if, for every point $p \in X$, there exists an involutive holomorphic isometry σ_p of X such that p is an isolated fixed point. An irreducible Hermitian symmetric spaces of noncompact type can be, by the Harish-Chandra embedding (cf. [M3], [Wo]), realized as a bounded domain in some complex Euclidean space. Such domains are convex, circular and sometimes are called bounded symmetric domains. Moreover, the boundary of a bounded symmetric domain D is non-smooth and contains complex analytic varieties, unless D is the unit ball. Among the irreducible bounded symmetric domains, there are so-called classical ones as opposed to two exceptional cases. The classical ones, sometimes referred as Cartan's classical domains, can be classified into four types (cf. [M3]). We will write $D^I_{p,q}, D^{II}_m, D^{III}_m, D^{IV}_m$ for the four types of classical domains, respectively. See Section 4 for their definitions and more detailed discussion. We prove in Section 4 that the smooth boundary of each classical domain, if it is not biholomorphic to the unit ball, must be uniformly $2-$nondegenerate (See also [KaZ] for related non-degeneracy results). The $2-$nondegeneracy was known to experts at least in some cases (for instance, the type IV case). In this paper, we give a complete treatment for all types in an explicit and computational way, with the help of the boundary orbit theorem for bounded symmetric domains (See [Wo]).

The complex unit ball $\mathbb{B}^n$ in the complex $n-$dimensional space is a special case of the type I classical domain. In this regard, the following propositions can be regarded as a natural extension of results of Forstnerič [Fr1] and Huang [Hu3] along the line of studying mappings into the boundaries of bounded symmetric domains.

Proposition 1.2. *Let $q \geq p \geq 2$ and M a strongly pseudoconvex real algebraic (smooth, real analytic respectively) real hypersurface in $\mathbb{C}^{p+q-1}$. Let F be a CR-transversal CR map of class C^2 from M to a smooth piece of the boundary of $D^I_{p,q}$. Then F extends to an algebraic holomorphic map (F is smooth, real analytic everywhere on M respectively).*

Proposition 1.3. *Let $m \geq 4$ and M a strongly pseudoconvex real algebraic (smooth, real analytic respectively) real hypersurface in $\mathbb{C}^{2m-3}$. Let F be a CR-transversal CR map of class C^2 from M to a smooth piece of the boundary of D^{II}_m. Then F extends to an algebraic holomorphic map (F is smooth, real analytic everywhere on M respectively).*

Proposition 1.4. *Let $m \geq 2$ and M a strongly pseudoconvex real algebraic (smooth, real analytic respectively) real hypersurface in $\mathbb{C}^m$. Let F be a CR-transversal CR map of class C^2 from M to a smooth piece of the boundary of D^{III}_m. Then F extends to an algebraic holomorphic map (F is smooth, real analytic everywhere on M respectively).*

Note in Propositions 1.2-1.4, the codimension can be arbitrarily large if we increase p, q and m. On the other hand, if we fix the target hypersurface (i.e., the smooth boundary of $D^I_{p,q}, D^{II}_m, D^{III}_m$, respectively) in Propositions 1.2-1.4, and replace the source M by a real hypersurface of lower dimension, then the regularity or algebraicity result may fail, even assuming higher initial regularity of the map. We illustrate this by explicit examples in Section 2. However, in sharp contrast to other types, we have the following result (See part (2)) for type IV domains.

Proposition 1.5. *(1). Let $m \geq 2$ and M a strongly pseudoconvex smooth (real analytic respectively) real hypersurface in $\mathbb{C}^m$. Let F be a CR-transversal CR map of class C^2 from M to a smooth piece of the boundary of D^{IV}_{m+1}. Then F is smooth (real analytic respectively) everywhere.*

(2). Let $m > n \geq 1$ and M a strongly pseudoconvex real algebraic real hypersurface in $\mathbb{C}^{n+1}$. Let F be a CR-transversal CR map of class C^{m-n+1} from M to a smooth piece of the boundary of D^{IV}_{m+1}. Then F extends to an algebraic holomorphic map.

Note the first part of Proposition 1.5 has already been established in [KLX].

Remark 1.6. If there is a transversal CR map F of class C^2 from a strongly pseudoconvex hypersurface M in $\mathbb{C}^n$ to a smooth piece of the boundary of $D^I_{p,q}$ (respectively, $D^{II}_m, D^{III}_m, D^{IV}_m$) then $n \leq p+q-1$ (respectively, $n \leq 2m-3; n \leq m; n \leq m-1$). See Proposition 4.1, 4.6, 4.7, 4.10 (See also Lemma 3 in [M5]). In the equality cases, the existence of such map F as described in Propositions 1.2-1.5 follows from the work of Mok [M5] when M is an open piece of the unit sphere. Indeed, F can be the boundary value of some holomorphic isometric maps from the unit ball to classical domains. See explicit formulas for such maps in [XY2].

The last part of the paper is devoted to studying proper maps into classical domains; in particular, we establish algebraicity result for proper mappings into type I and IV domains.

Theorem 2. *Let $\Omega \subset \mathbb{C}^{q+1}(q \geq 2)$ be a smoothly bounded domain with real algebraic boundary. Then any holomorphic proper map F from Ω to $D^I_{2,q}$, that is C^2-smooth up to some open piece of the boundary $\partial\Omega$, must be algebraic.*

Note in the above theorem, the codimension of the source and target domains equals to $q-1$, which can be arbitrarily large, while the initial regularity assumption for the map is always C^2. It is not clear to us whether this C^2 smoothness assumption is optimal. However, at least some initial boundary regularity has to be assumed to expect the algebraicity in Theorem 2. See Remark 1.9.

Remark 1.7.
- In Theorem 2, we can say more about the map: F extends holomorphically across a dense open subset G of $\partial\Omega$ and maps G to the smooth piece of $\partial D^I_{2,q}$. But in general, unlike rational proper maps between balls (cf. [CS]), one cannot expect G to be the whole boundary $\partial\Omega$ even when Ω is the unit ball $\mathbb{B}^{q+1}$. See the map F in Example 2.2 or [XY2].
- If we fix the target $D^I_{2,q}$ in Theorem 2, and replace the source Ω by a lower-dimensional domain, the algebraicity result may fail(no matter what initial regularity of the map is assumed). See Example 2.1.

By a result of Dor (See Theorem 1 in [Do]. See also [DF]), for any convex open set D in $\mathbb{C}^{N+1}(N \geq n \geq 2)$, there exists a proper map from the unit ball $\mathbb{B}^n$ to D. As bounded symmetric domains are all convex, this indicates the unit ball $\mathbb{B}^n(n \geq 2)$ can be properly mapped to any classical domain of higher dimension by a holomorphic map. We observe that $\mathbb{B}^{q+1}(q \geq 2)$ is indeed the maximal dimensional ball that admits a holomorphic proper map to $D^I_{2,q}$ with C^2 extension up to some boundary point.

Corollary 1.8. *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain with real analytic boundary, F a holomorphic proper map from Ω to $D^I_{2,q}(n \geq q+2 \geq 4)$. Then F cannot be extended C^2 smoothly up to any open piece of $\partial\Omega$.*

Remark 1.9. In particular, by Corollary 1.8 and [Do], there exists a non-algebraic proper holomorphic map from $\mathbb{B}^{q+2}$ to $D^I_{2,q}$. By slicing $\mathbb{B}^{q+2}$, we can get a non-algebraic proper holomorphic map from $\mathbb{B}^{q+1}$ to $D^I_{2,q}$ (This map is not C^2 at any boundary point by Theorem 2). This indicates at least some initial regularity is needed to conclude the algebraicity in Theorem 2.

Theorem 3. *Let $\Omega \subset \mathbb{C}^{n+1}, n \geq 1$, be a smoothly bounded domain with real algebraic boundary. Then any holomorphic proper map F from Ω to $D^{IV}_{m+1}(m > n)$, that is C^{m-n+1} smooth up to some boundary point of Ω, must be algebraic.*

Remark 1.10. Moreover, in Theorem 3, F extends holomorphically across a dense open subset G of $\partial\Omega$ and maps G to the smooth piece of ∂D^{IV}_{m+1}. As in Remark 1.7, we cannot expect G to be the whole boundary of Ω even if Ω is the unit ball.

As an application of Theorem 3, we can drop the transversality assumption and weaken the boundary regularity assumption of the rigidity theorem in [XY1].

Corollary 1.11. *Let F be a holomorphic proper map from $\mathbb{B}^n(5 \leq n+1 \leq m \leq 2n-3)$ to the Lie ball D^{IV}_m that is $C^{m-n+1}-$smooth up to some open piece of $\partial\mathbb{B}^n$. Then F is an isometric map with respect to the Bergman metrics. Moreover, when $m = n+1$, after composing appropriate automorphisms of $\mathbb{B}^n$ and D^{IV}_{n+1}, F is one of the following two maps:*

$$\left(z_1, \cdots, z_{n-1}, \frac{\frac{1}{2}\sum_{i=1}^{n-1} z_i^2 - z_n^2 + z_n}{\sqrt{2}(1-z_n)}, \sqrt{-1}\frac{\frac{1}{2}\sum_{i=1}^{n-1} z_i^2 + z_n^2 - z_n}{\sqrt{2}(1-z_n)}\right); \tag{1.1}$$

or

$$\left(z_1, \cdots, z_{n-1}, z_n, 1 - \sqrt{1 - \sum_{j=1}^{n} z_j^2}\right). \tag{1.2}$$

The paper is organized as follows. In Section 2, we introduce some preliminaries on classical domains and some basic notions in CR geometry. We also provide various explicit examples of mapping into classical domains to support some remarks in the introduction. Section 3 is devoted to the proof of Theorem 1. We investigate the CR geometric structure of smooth boundaries of classical domains in Section 4. As applications, we use it to prove Propositions 1.2-1.5. Theorem 2 and Theorem 3, as well as Corollary 1.8, will be proved in Section 5.

Acknowledgment: The author thanks Peter Ebenfelt, Xiaojun Huang, Nordine Mir and Yuan Yuan for helpful comments. He also thanks Sui-Chung Ng for valuable discussions on Hermitian symmetric spaces.

2. Preliminaries and some examples

2.1. **Classical domains and examples.** The rank r of a bounded symmetric domain D can be defined as the dimension of the maximal polydisc that can be totally geodesically embedded into D. For an irreducible bounded symmetric domain D in $\mathbb{C}^n$, the boundary ∂D decomposes into exactly r orbits under the action of the identity component G_0 of $\mathrm{Aut}(D)$: $\partial D = \cup_{i=1}^r E_i$. Here E_1 is the unique open orbit, which is indeed the smooth part of ∂D (cf. Lemma 2.2.3 in [MN]). Moreover, E_r is the Shilov boundary, and E_k lies in the closure of E_l if $k > l$. As mentioned in the introduction, irreducible bounded symmetric domains can be classified as Cartan's four types of classical domains and two exceptional cases (cf. [M3], [Wo]).

Assume $p \leq q$ and write $\mathbb{C}^{p\times q}$ as the space of $p \times q$ matrices with entries of complex numbers. The classical domain of type I is defined by:

$$D^I_{p,q} = \{Z \in \mathbb{C}^{p\times q} : I_p - Z\overline{Z}^t > 0\}.$$

The boundary of $D^I_{p,q}$ is given

$$\partial D^I_{p,q} = \{Z \in \mathbb{C}^{p\times q} : I_p - Z\overline{Z}^t \geq 0; \det(I_p - Z\overline{Z}^t) = 0\}.$$

Denote by $\mathbb{C}_{II}^{\frac{m(m-1)}{2}} = \{Z \in \mathbb{C}^{m\times m} : Z = -Z^t\}$ and $\mathbb{C}_{III}^{\frac{m(m+1)}{2}} = \{Z \in \mathbb{C}^{m\times m} : Z = Z^t\}$ the set of all skew-symmetric and symmetric square matrices of size $m \times m$, respectively. The type II and type

III classical domains are submanifolds of $D^I_{m,m}$ defined as:

$$D^{II}_m = \{Z \in \mathbb{C}_{II}^{\frac{m(m-1)}{2}} : I_m - Z\overline{Z}^t > 0\};$$

$$D^{III}_m = \{Z \in \mathbb{C}_{III}^{\frac{m(m+1)}{2}} : I_m - Z\overline{Z}^t > 0\}.$$

Note D^{II}_3 is biholomorphic to the unit ball $\mathbb{B}^3$. The boundaries of D^{II}_m and D^{III}_m are given by the following formulas:

$$\partial D^{II}_m = \{Z \in \mathbb{C}_{II}^{\frac{m(m-1)}{2}} : I_m - Z\overline{Z}^t \geq 0; \det(I_m - Z\overline{Z}^t) = 0\};$$

$$\partial D^{III}_m = \{Z \in \mathbb{C}_{III}^{\frac{m(m+1)}{2}} : I_m - Z\overline{Z}^t \geq 0; \det(I_m - Z\overline{Z}^t) = 0\}.$$

The type IV classical domain, often called the Lie ball, is defined by

$$D^{IV}_m = \{Z = (z_1, \cdots, z_m) \in \mathbb{C}^m : Z\overline{Z}^t < 2,\ 1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2 > 0\}.$$

Note D^{IV}_2 is biholomorphic to the bidisc. The boundary of D^{IV}_m is given by

$$\partial D^{IV}_m = \{Z = (z_1, \cdots, z_m) \in \mathbb{C}^m : Z\overline{Z}^t \leq 2, 1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2 = 0\}.$$

We will discuss more details about the boundaries of classical domains in Section 4. We next give some examples of proper maps into classical domains. Let $l, k \in \mathbb{Z}, l \geq 2, k \geq 1$. By Theorem 2.7 in [BX2], there exists a holomorphic function ϕ in $\mathbb{B}^l$ which extends C^k smoothly to the sphere $\partial\mathbb{B}^l$ but not C^{k+1} up to any open piece of $\partial\mathbb{B}^l$. Set $\mathcal{F}(l, k)$ to be the collections of all such functions ϕ.

In the following examples, we always equip an irreducible bounded symmetric domain D with the normalized Bergman metric (Kähler-Einstein metric) such that the minimal disc is of constant Gaussian curvature -2.

Example 2.1. Let $H : \mathbb{B}^{q-1} \to D^I_{2,q}, q \geq 3$, be defined by

$$H(z_1, \cdots, z_{q-1}) = \begin{pmatrix} z_1 & \cdots & z_{q-1} & 0 \\ 0 & \cdots & 0 & \phi \end{pmatrix}.$$

Fix $k \geq 1$. Let $\phi \in \mathcal{F}(q-1, k)$. We can additionally assume $|\phi| < 1$ in $\overline{\mathbb{B}^{q-1}}$. Then H is a holomorphic proper map and has C^k extension up to $\partial\mathbb{B}^{q-1}$ but is not C^{k+1} up to any boundary point. Moreover, it is easy to see H maps $\partial\mathbb{B}^{q-1}$ to the smooth part of $\partial D^I_{2,q}$ (See subsection 4.1). Clearly the conclusion in Theorem 2 fails in this case.

Example 2.2. Let $q \geq p \geq 3$. Write the coordinates in $\mathbb{B}^{p+q-3}$ as $z = (z_1, \cdots, z_{p-1}, w_2, \cdots, w_{q-1})$. Let $F : \mathbb{B}^{p+q-3} \to D^I_{p-1,q-1}$ be an isometric map with respect to the normalized Bergman metrics. The existence of such a map was proved in [M5]. An explicit example was given in [XY2] (See equation (42) in [XY2]):

$$F(z)=\begin{pmatrix} z_1 & z_2 & \cdots & z_{q-1} \\ w_2 & f_{22} & \cdots & f_{2(q-1)} \\ \cdots & \cdots & \cdots & \cdots \\ w_{p-1} & f_{(p-1)2} & \cdots & f_{(p-1)(q-1)} \end{pmatrix},$$

where $f_{ij}=\frac{w_i z_j}{z_1-1}, 2\leq i\leq p-1, 2\leq j\leq q-1$. Indeed, it was proved in [XY2] that F satisfies

$$\det(I_p-F\overline{F}^t)=1-\sum_{j=1}^{q-1}|z_i|^2-\sum_{i=2}^{p-1}|w_i|^2.$$

By (the proof on page 9 of) Theorem 2 in [M5], F maps a generic point on $\partial\mathbb{B}^{p+q-3}$ to a smooth point of $\partial D^I_{p-1,q-1}$. Fix $k\geq 1$. Let $\phi\in\mathcal{F}(p+q-3,k)$ with $|\phi|<1$ on $\overline{\mathbb{B}^{p+q-3}}$. Set the map $H:\mathbb{B}^{p+q-3}\to D^I_{p,q}$ to be

$$H(z)=\begin{pmatrix} F(z) & \mathbf{0}^t_{p-1} \\ \mathbf{0}_{q-1} & \phi(z) \end{pmatrix}$$

with $\mathbf{0}_{p-1}$ and $\mathbf{0}_{q-1}$ the $(p-1)-$dimensional and $(q-1)-$dimensional zero row vector, respectively. Then H gives a holomorphic proper map and is C^k-smooth up to a generic boundary point. Moreover, by the propery of F and the fact that $|\phi|<1$, H maps a generic point on the sphere to a smooth point of $\partial D^I_{p,q}$. However, H is not C^{k+1} at any boundary point.

Example 2.3. Let F be a rational holomorphic isometry from $\mathbb{B}^{2m-7}$ to $D^{II}_{m-2}(m\geq 6)$ with respect to the normalized Bergman metrics (the existence of such an isometry was proved in [M5] and an explicit example was given by (50) in [XY2]). We define $H:\mathbb{B}^{2m-7}\to D^{II}_m$ as

$$H(z)=\begin{pmatrix} F(z) & \mathbf{0}^t & \mathbf{0}^t \\ \mathbf{0} & 0 & \phi(z) \\ \mathbf{0} & -\phi(z) & 0 \end{pmatrix},$$

where $\mathbf{0}$ is a $(m-2)-$dimensional zero row vector, $\phi\in\mathcal{F}(2m-7,k)$ with $|\phi|<1$ on $\overline{\mathbb{B}^{2m-7}}$. It follows from Theorem 2 of [M5] that F extends holomorphically across a generic point on the sphere and maps it to a smooth point of ∂D^{II}_{m-2}. Thus H is C^k-smooth up to a generic boundary point and also maps it to a smooth point of ∂D^{II}_m, but H is not C^{k+1} at any boundary point.

Example 2.4. Let F be a rational holomorphic isometry from $\mathbb{B}^{m-1}$ to $D^{III}_{m-1}(m\geq 3)$ (the existence of such isometry was proved in [M5] and an explicit example was given by equation (53) in [XY2]). We define $H:\mathbb{B}^{m-1}\to D^{III}_m$ to be:

$$H(z)=\begin{pmatrix} F(z) & \mathbf{0}^t \\ \mathbf{0} & \phi(z) \end{pmatrix}.$$

Here $\mathbf{0}$ is a $m-$dimensional zero row vector, $\phi\in\mathcal{F}(m-1,k)$ with $|\phi|<1$ on $\overline{\mathbb{B}^{m-1}}$. It follows [M5] that F maps a generic point on the sphere to a smooth point of ∂D^{III}_{m-1}. Consequently, H has a C^k extension to a generic point on $\partial\mathbb{B}^{m-1}$ and maps it to a smooth point of ∂D^{III}_m. However, H is not

C^{k+1} at any boundary point. Hence the conclusions in Theorem 1 fail in this case. By Proposition 4.7, the smooth boundary of ∂D_m^{III} is uniformly $2-$nondegenerate and the Levi form has $m-1$ nonzero eigenvalues of the same sign. In particular, it has more nonzero eigenvalues than the Levi form of $\partial\mathbb{B}^{m-1}$. This supports the assertion in Remark 1.1.

2.2. **Two notions of degeneracy.** In this section, we recall various notions of degeneracy in CR geometry, and their relations. The following definition was introduced in [BHR].

Definition 2.5. Let M be a smooth generic submanifolds in $\mathbb{C}^N$ of CR-dimension d and CR-codimension n, and $p \in M$. Let $\rho = (\rho_1, ..., \rho_d)$ be the defining function of M near p, and choose a basis $L_1, ..., L_n$ of CR vector fields near p. For a multiindex $\alpha = (\alpha_1, ..., \alpha_n)$, write $L^\alpha = L_1^{\alpha_1} ... L_n^{\alpha_n}$. Define the increasing sequence of subspaces $E_l(p) (0 \le l \le k)$ of $\mathbb{C}^N$ by

$$E_l(p) = \mathrm{Span}_{\mathbb{C}}\{L^\alpha \rho_{\mu,Z}(Z, \overline{Z})|_{Z=p} : 0 \le |\alpha| \le l, 1 \le \mu \le d\}.$$

Here $\rho_{\mu,Z} = (\frac{\partial\rho_\mu}{\partial z_1}, \cdots, \frac{\partial\rho_\mu}{\partial z_N})$, and $Z = (z_1, \cdots, z_N)$ are the coordinates in $\mathbb{C}^N$. We say that M is $k-$*nondegenerate at* p, $k \ge 1$ if

$$E_{k-1}(p) \ne E_k(p) = \mathbb{C}^N.$$

We say M is $k-$degenerate at p if $E_k(p) \ne \mathbb{C}^N$.

We say M is (everywhere) finitely nondegenerate if M is $k(p)-$nondegenerate at every $p \in M$ for some integer $k(p)$ depending on p. A smooth CR-manifold M of hypersurface type is Levi-nondegenerate at $p \in M$ if and only if M is $1-$nondegenerate at p. A real hypersurface is called uniformly $2-$nondegenerate if it is $2-$nondegenerate at every point. This notion of degeneracy is then generalized to CR-mappings by Lamel [La1] as follows.

Definition 2.6. Let $M \subset \mathbb{C}^N, M' \subset \mathbb{C}^{N'}$ be two generic CR-submanifolds of CR dimension n, n', respectively. Let $H : M \to M'$ be a CR-mapping of class C^k near $p_0 \in M$. Let $\rho = (\rho_1, \cdots, \rho_{d'})$ be local defining functions for M' near $H(p_0)$, and choose a basis $L_1, \cdots, L_n$ of CR vector fields for M near p_0. If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multiindex, write $L^\alpha = L_1^{\alpha_1} \cdots L_n^{\alpha_n}$. Define the increasing sequence of subspaces $E_l(p_0) (0 \le l \le k)$ of $\mathbb{C}^{N'}$ by

$$E_l(p_0) = \mathrm{Span}_{\mathbb{C}}\{L^\alpha \rho_{\mu,Z'}(H(Z), \overline{H(Z)})|_{Z=p_0} : 0 \le |\alpha| \le l, 1 \le \mu \le d'\}.$$

Here $\rho_{\mu,Z'} = (\frac{\partial\rho_\mu}{\partial z'_1}, \cdots, \frac{\partial\rho_\mu}{\partial z'_{N'}})$, and $Z' = (z'_1, \cdots, z'_{N'})$ are the coordinates in $\mathbb{C}^{N'}$. We say that H *is* k_0-*nondegenerate at* p_0 $(0 \le k_0 \le k)$ if

$$E_{k_0-1}(p_0) \ne E_{k_0}(p_0) = \mathbb{C}^{N'}.$$

A manifold M is k_0-nondegenerate if and only if the identity map from M to M is k_0-nondegenerate.

2.3. **Normalization of the map.** We will need an auxiliary normalization result (Proposition 2.7) for CR-maps from a strongly pseudoconvex real hypersurface to a pseudoconvex Levi degenerate hypersurface. It is Proposition 3.4 from [KLX]. In [KLX], it did not give a proof and only indicated it can be shown similarly as Proposition 3.3 there. For self-containedness, we will sketch a proof in this paper.

Proposition 2.7. *Let $M \subset \mathbb{C}^{n+1}(n \geq 1)$ be a strongly pseudoconvex real-analytic (resp. smooth) real hypersurface, $M' \subset \mathbb{C}^{N+1}(N > n)$ a real-analytic (resp. smooth) real hypersurface. Assume that $F = (F_1, ..., F_{N+1}) : M \mapsto M'$ is a CR-transversal CR-mapping of class C^2 near $p_0 \in M$ with $F(p_0) = q_0$, and that the Levi form of M' has exactly $m(n \leq m \leq N)$ eigenvalues of the same sign at q_0. Then, after appropriate holomorphic changes of coordinates in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$ respectively, we have $p_0 = 0, q_0 = 0$, and the following normalization hold: M is defined by*

$$r(Z, \overline{Z}) = -\mathrm{Im} z_{n+1} + \sum_{i=1}^{n} |z_i|^2 + \psi(Z, \overline{Z}), \quad \psi = O(|Z|^3) \tag{2.1}$$

near 0, and M' is defined by

$$\rho(W, \overline{W}) = -\mathrm{Im} w_{N+1} + \sum_{j=1}^{m} |w_j|^2 + \phi(W, \overline{W}), \quad \phi = O(|W|^3) \tag{2.2}$$

near 0, where $Z = (z_1, ..., z_{n+1}), W = (w_1, ..., w_{N+1})$ are the coordinates of $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$, respectively. Here ψ and ϕ are real-analytic (resp. smooth). Furthermore, F satisfies:

$$\frac{\partial F_i}{\partial z_j}(0) = \delta_{ij}\sqrt{\lambda},\ 1 \leq i, j \leq n; \quad \frac{\partial F_{N+1}}{\partial z_{n+1}}(0) = \lambda \tag{2.3}$$

for some $\lambda > 0$, and moreover,

$$\frac{\partial F_k}{\partial z_j}(0) = 0, 1 \leq j \leq n,\ n+1 \leq k \leq N; \tag{2.4}$$

$$\frac{\partial F_{N+1}}{\partial z_j}(0) = 0, 1 \leq j \leq n. \tag{2.5}$$

Proof. We assume, after a holomorphic change of coordinates in $\mathbb{C}^{n+1}$, $p_0 = 0$ and that M is defined near 0 by (2.1). After a holomorphic change of coordinates in $\mathbb{C}^{N+1}$, we assume that $q_0 = F(p_0) = 0$ and that M' is locally defined near 0 by (2.2). Then F satisfies:

$$-\frac{F_{N+1} - \overline{F_{N+1}}}{2i} + \sum_{i=1}^{m} |F_i|^2 + \phi(F, \overline{F}) = 0, \tag{2.6}$$

along M. Since F is CR-transversal, we get $\lambda := \frac{\partial F_{N+1}}{\partial s}|_0 \neq 0$, where we write $z_{n+1} = s + it$(cf. [BER2]). Moreover, (2.6) shows that the imaginary part of F_{N+1} vanishes to second order at the origin, and so λ is real. Write a basis $\{L_j\}_{1 \leq j \leq n}$ for the CR vector fields along M near p_0 as $L_j = 2i\left(\frac{\partial r}{\partial z_{n+1}}\frac{\partial}{\partial z_j} - \frac{\partial r}{\partial z_j}\frac{\partial}{\partial z_{n+1}}\right), 1 \leq j \leq n$. By applying $\overline{L_j}, \overline{L_j L_k}, 1 \leq j, k \leq n$ to the equation (2.6) and evaluating at 0, we get: $\frac{\partial F_{N+1}}{\partial z_j}(0) = 0$, $\frac{\partial^2 F_{N+1}}{\partial z_j \partial z_k}(0) = 0, 1 \leq j, k \leq n$.

Hence we have,

$$F_{N+1}(Z) = \lambda z_{n+1} + O(|\widetilde{z}||z_{n+1}| + |z_{n+1}|^2) + o(|Z|^2). \tag{2.7}$$

For $1 \leq j \leq N$, we write $F_j = a_j z_{n+1} + \sum_{i=1}^{n} a_{ij} z_i + O(|Z|^2)$, for some $a_j \in \mathbb{C}, a_{ij} \in \mathbb{C}, 1 \leq i \leq n-1, 1 \leq j \leq n$. Or equivalently,

$$(F_1, ..., F_N) = z_{n+1}(a_1, ..., a_N) + (z_1, ..., z_n)A + (\hat{F}_1, ..., \hat{F}_N), \tag{2.8}$$

where $A = (a_{ij})_{1\leq i\leq n, 1\leq j\leq N}$ is an $n \times N$ matrix, and $\hat{F}_j = O(|Z|^2), 1 \leq j \leq N$. We substitute (2.7) and (2.8) into (2.6) to get,

$$\lambda|\widetilde{z}|^2 + O(|\widetilde{z}||z_{n+1}| + |z_{n+1}|^2) + o(|Z|^2) = \widetilde{z} A D_m A^* \overline{\widetilde{z}}^t + O(|\widetilde{z}||z_{n+1}| + |z_{n+1}|^2) + o(|Z|^2), \tag{2.9}$$

Here write $\widetilde{z} = (z_1, ..., z_n)$ and D_m is the diagonal matrix whose first m diagonal entries are 1 and the rest are 0. Equip $\widetilde{z}$ with weight 1, and z_{n+1} with weight 2. Compare terms with weight 2 at both sides of (2.9) to get:

$$\lambda \mathbf{I}_n = A D_m A^*. \tag{2.10}$$

It follows that $\lambda > 0$. Write $A = (B_0, \mathbf{b})$, where B_0 is an $n \times m$ matrix, $\mathbf{b}$ is an $n \times (N-m)$ matrix. Note (2.10) yields that $B_0 \overline{B_0}^t = \lambda \mathbf{I}_n$. We can then add more rows to B_0 and get an $m \times m$ matrix B such that $B\overline{B}^t = \lambda I_m$. We now apply the following holomorphic change of coordinates: $\widetilde{W} = WD$ or $W = \widetilde{W} D^{-1}$, where we set

$$D = \begin{pmatrix} \frac{1}{\sqrt{\lambda}} \overline{B}^t & \mathbf{c} & \mathbf{0} \\ \mathbf{0}_{(N-m)\times m} & \mathbf{I}_{(N-m)\times(N-m)} & 0 \\ \mathbf{0}^t & 0 & 1 \end{pmatrix},$$

and $\mathbf{0}$ is the $m-$dimensional zero column vector, $\mathbf{c}$ is an $m \times (N-m)$ matrix to be determined. We compute

$$D^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\lambda}} B & \mathbf{d} & \mathbf{0} \\ \mathbf{0}_{(N-m)\times m} & \mathbf{I}_{(N-m)\times(N-m)} & 0 \\ \mathbf{0}^t & 0 & 1 \end{pmatrix},$$

where $\mathbf{d} = -\frac{1}{\sqrt{\lambda}} B\mathbf{c}$. We write the new defining function of M' and the map as $\widetilde{\rho}$ and $\widetilde{F} = (\widetilde{F}_1, ..., \widetilde{F}_{N+1})$ in the new coordinates $\widetilde{W} = (\widetilde{w}_1, ..., \widetilde{w}_{N+1})$, respectively. It is easy to see (See Lemma 3.1 in [KXL]) that $\widetilde{\rho}$ still has the form of (2.2). More precisely, $\widetilde{\rho}(\widetilde{W}, \overline{\widetilde{W}}) = -\mathrm{Im}\widetilde{w}_{N+1} + \sum_{j=1}^{m} |\widetilde{w}_j|^2 + \widetilde{\phi}(\widetilde{W}, \overline{\widetilde{W}})$, where $\widetilde{\phi} = O(|\widetilde{W}|^3)$ is also a real-analytic (resp. smooth) function defined near 0. Moreover, since $\widetilde{F} = FD$, it is easy to see that

$$\frac{\partial \widetilde{F}_i}{\partial z_j}(0) = \delta_{ij}\sqrt{\lambda}, \ 1 \leq i, j \leq n.$$

$$\frac{\partial \widetilde{F}_i}{\partial z_j}(0) = 0, \ n+1 \leq i \leq m, 1 \leq j \leq n.$$

Here we denote by δ_{ij} the Kronecker symbol that takes value 1 when $i = j$ and 0 otherwise.

Lemma 2.8. *We can choose an appropriate* $\mathbf{c}$ *such that*

$$\frac{\partial \widetilde{F}_k}{\partial z_j}(0) = 0, 1 \leq j \leq n, \ m+1 \leq k \leq N. \tag{2.11}$$

Proof: Note that $(\widetilde{F}_{m+1}, \cdots, \widetilde{F}_N) = (F_1, ..., F_N)\begin{pmatrix} \mathbf{c} \\ \mathbf{I}_{(N-m)\times(N-m)} \end{pmatrix}$. Combining this with (2.8), we obtain (2.11) holds if and only if $A\begin{pmatrix} \mathbf{c} \\ \mathbf{I}_{(N-m)\times(N-m)} \end{pmatrix} = \mathbf{0}_{n\times(N-m)}$. Recall $A = (B_0, \mathbf{b})$. We can choose $\mathbf{c}$ such that $B_0\mathbf{c} = -b$ as B_0 is of full rank. ■

Proposition 2.7 follows if we still write W, F and ρ instead of $\widetilde{W}, \widetilde{F}$ and $\widetilde{\rho}$. □

3. Proof of Theorem 1

We will prove the two parts of Theorem 1 in separated subsections.

3.1. **Proof of Theorem 1.** In this subsection we give a proof for Theorem 1. The proof relies on a special nondegeneracy property of the cubic tensor(defined using commutators of three vector fields, cf. [E1]) for the uniformly $2-$nondegenerate hypersurface M' where the Levi form is of constant rank. This approach is inspired by [E1] and the readers are referred to the paper for more results on uniformly Levi-degenerate hypersurfaces. To adapt with the normalization (Proposition 2.7), we will make the above idea explicit by computations in local coordinates.

Fix $p_0 \in M$ and write $q_0 = F(p_0) \in M'$. Choose appropriate coordinates such that $p_0 = 0, q_0 = 0$ and the normalization in Proposition 2.7 holds. Note under the assumption of Theorem 1, $m = n$ in (2.2). Let r, ψ, ρ, ϕ be as in (2.1) and (2.2). We will write for $1 \leq k \leq N$,

$$\Lambda_k = 2i\left(\frac{\partial \rho}{\partial \overline{w}_{N+1}}\frac{\partial}{\partial \overline{w}_k} - \frac{\partial \rho}{\partial \overline{w}_k}\frac{\partial}{\partial \overline{w}_{N+1}}\right), \tag{3.1}$$

where $\{\Lambda_k\}_{1\leq k\leq N}$ forms a basis for the CR vector fields along M' near 0. Note by (2.2)

$$\begin{aligned} \Lambda_j &= (1 + 2i\phi_{\overline{N+1}})\frac{\partial}{\partial \overline{w}_j} - 2i(w_j + \phi_{\overline{j}})\frac{\partial}{\partial \overline{w}_{N+1}}, \text{ if } 1 \leq j \leq n, \\ \Lambda_k &= (1 + 2i\phi_{\overline{N+1}})\frac{\partial}{\partial \overline{w}_k} - 2i(\phi_{\overline{k}})\frac{\partial}{\partial \overline{w}_{N+1}}, \text{ if } n+1 \leq k \leq N. \end{aligned} \tag{3.2}$$

Here and in the following, we write for $1 \leq i, j, k \leq N+1$, $\phi_{\overline{i}} = \phi_{\overline{w}_i} = \frac{\partial \phi}{\partial \overline{w}_i}, \phi_i = \phi_{w_i} = \frac{\partial \phi}{\partial w_i}, \phi_{i\overline{j}} = \phi_{w_i\overline{w}_j} = \frac{\partial^2 \phi}{\partial w_i \partial \overline{w}_j}, \phi_{\overline{i}\overline{j}k} = \phi_{\overline{w}_i\overline{w}_j w_k} = \frac{\partial^3 \phi}{\partial \overline{w}_i \partial \overline{w}_j \partial w_k}$, etc.

Recall our notation $\rho_W := (\frac{\partial \rho}{\partial w_1}, ..., \frac{\partial \rho}{\partial w_{N+1}})$. We compute

$$\rho_W(W, \overline{W}) = (\overline{w}_1 + \phi_1, ..., \overline{w}_n + \phi_n, \phi_{n+1}, \cdots, \phi_N, \frac{i}{2} + \phi_{N+1}) \tag{3.3}$$

We thus have

$$\Lambda_1 \rho_W(W, \overline{W}) = \left(h_{11}, ..., h_{1(N+1)}\right), \tag{3.4}$$

where

$$
\begin{aligned}
h_{11} &= (1 + 2i\phi_{\overline{(N+1)}})(1 + \phi_{1\bar{1}}) - 2i(w_1 + \phi_{\bar{1}})\phi_{1\overline{(N+1)}}, \\
h_{1k} &= (1 + 2i\phi_{\overline{(N+1)}})\phi_{k\bar{1}} - 2i(w_1 + \phi_{\bar{1}})\phi_{k\overline{(N+1)}}, \text{ if } 2 \le k \le N+1.
\end{aligned}
\tag{3.5}
$$

Hence

$$\Lambda_1 \rho_W(W, \overline{W}) = (1 + O(1), O(1), ..., O(1)). \tag{3.6}$$

Here we write $O(m) = O(|W|^m)$ for any $m \ge 0$. Similarly, we have for $1 \le j \le n$,

$$\Lambda_j \rho_W(W, \overline{W}) = (O(1), ..., O(1), 1 + O(1), O(1), ..., O(1)), \tag{3.7}$$

where the term $1 + O(1)$ is at the j^{th} position. When $n + 1 \le k \le N$,

$$\Lambda_k \rho_W(W, \overline{W}) = (O(1), ..., O(1), \phi_{\overline{k}(n+1)} + O(2), \cdots, \phi_{\overline{k}N} + O(2), O(1)), \tag{3.8}$$

where the first n components are all $O(1)$.

For $n + 1 \le k \le N$, we write the $(n + 2) \times N$ matrix

$$\Delta_k := \begin{pmatrix} \rho_W \\ \Lambda_1 \rho_W \\ ... \\ \Lambda_n \rho_W \\ \Lambda_k \rho_W \end{pmatrix}. \tag{3.9}$$

Recall that the Levi form of M' has exactly n nonzero eigenvalues. Consequently, $\dim E_1(q) = n+1$ for any q near 0 (See Definition 2.6). This implies Δ_k has rank $n + 1$ near 0 along M'. Hence every $(n + 2) \times (n + 2)$ submatrix of Δ_k always has zero determinant near 0 along M'. We will write $\Delta_k (1, \cdots, n, l, N)$ as the submatrix of Δ_k formed by the $1^{\text{st}}, \cdots, n^{\text{th}}, l^{\text{th}}, (N + 1)^{\text{th}}$ columns, where $n + 1 \le l \le N$. By equations (3.3), (3.7) and (3.8), we conclude that the determinant of $\Delta_k (1, \cdots, n, l, N)$ equals

$$\pm \frac{i}{2} \phi_{\overline{k}l} + O(2). \tag{3.10}$$

Note (3.10) vanishes identically near 0 along M'. By applying $\Lambda_j, 1 \le j \le N$ to (3.10) and evaluating at 0, we obtain $\phi_{\overline{jk}l}(0) = 0$ for any $1 \le j \le N, n+1 \le k, l \le N$. Combining this with (3.8), we obtain

$$\Lambda_j \Lambda_k \rho_W(0) \in \mathrm{Span}_{\mathbb{C}}\{\rho_W, \Lambda_1 \rho_W, \cdots, \Lambda_n \rho_W\}|_{W=0} \text{ if } 1 \le j \le N, n+1 \le k \le N. \tag{3.11}$$

By the fact that M' is 2-nondegenerate at 0, we have:

$$\mathrm{Span}_{\mathbb{C}}\{\Lambda^\alpha \rho_W(W, \overline{W}) : 0 \le |\alpha| \le 2\}|_{W=0} = \mathbb{C}^{N+1}. \tag{3.12}$$

We combine (3.11), (3.12) to obtain

$$\mathrm{Span}_{\mathbb{C}}\{\rho_W, \Lambda_1 \rho_W, \cdots, \Lambda_n \rho_W, \{\Lambda_j \Lambda_k \rho_W\}_{1 \le j,k \le n}\}|_{W=0} = \mathbb{C}^{N+1}. \tag{3.13}$$

We now need the following lemma. Recall we choose a basis of CR vector fields along M near $p_0 = 0 : L_j = 2i \left(\frac{\partial r}{\partial \overline{z}_{n+1}} \frac{\partial}{\partial \overline{z}_j} - \frac{\partial r}{\partial \overline{z}_j} \frac{\partial}{\partial \overline{z}_{n+1}} \right), 1 \le j \le n.$

Lemma 3.1. *The following equations hold at* 0 :

$$\rho_W(F,\overline{F})|_{Z=0} = \rho_W(W,\overline{W})|_{W=0}; \tag{3.14}$$

$$L_j\rho_W(F,\overline{F})|_{Z=0} = \sqrt{\lambda}\cdot\Lambda_i\rho_W(W,\overline{W})|_{W=0}, \quad \textit{for} \quad 1\leq j\leq n; \tag{3.15}$$

For $1\leq j,k\leq n$, *write* $\mathbf{S}_{jk}$ *and* $\mathbf{P}_{jk}$ *as the last* $N-n+1$ *components of* $L_jL_k\rho_W(F,\overline{F})|_{Z=0}$ *and* $\Lambda_j\Lambda_k\rho_W(W,\overline{W})|_{W=0}$, *respectively. Then*

$$\mathbf{S}_{jk} = \lambda\mathbf{P}_{jk}, \quad \textit{for} \quad 1\leq j,k\leq n. \tag{3.16}$$

Proof of Lemma 3.1: With the normalization in Proposition 2.7 and equations (3.7), (3.8), an easy computation yields that,

$$\rho_W(F,\overline{F})(0) = (0,...,0,\frac{i}{2}); \tag{3.17}$$

$$L_j\rho_W(F,\overline{F})(0) = (0,...,0,\sqrt{\lambda},0,...,0), 1\leq j\leq n. \tag{3.18}$$

Here $\sqrt{\lambda}$ is at the j^{th} position. We thus obtain equations (3.14), (3.15) by comparing the above equations to (3.3), (3.7). To prove (3.16), we note that for fixed $1\leq j_0,k_0\leq n$,

$$\Lambda_{j_0}\Lambda_{k_0}\rho_W(W,\overline{W})|_{W=0} = \left(\phi_{\overline{j_0k_0}1}(0),\cdots,\phi_{\overline{j_0k_0}(N+1)}(0)\right).$$

On the other hand, if we write

$$L_{j_0}L_{k_0}\rho_W(F,\overline{F})(0) := (\nu_1,\cdots,\nu_{N+1}), \tag{3.19}$$

then for each $n+1\leq l\leq N+1$, we have by (3.3), (2.3), (2.4),

$$\nu_l = \frac{\partial^2\phi_l(F,\overline{F})}{\partial\overline{z}_{j_0}\partial\overline{z}_{k_0}}\bigg|_0 = \sum_{i,j=1}^{N+1}\frac{\partial^2\phi_l}{\partial\overline{w}_i\partial\overline{w}_j}\bigg|_0\overline{\frac{\partial F_i}{\partial z_{j_0}}}\bigg|_0\overline{\frac{\partial F_j}{\partial z_{k_0}}}\bigg|_0 = \phi_{\overline{j_0k_0}l}(0)\overline{\frac{\partial F_{j_0}}{\partial z_{j_0}}}\bigg|_0\overline{\frac{\partial F_{k_0}}{\partial z_{k_0}}}\bigg|_0 = \lambda\phi_{\overline{j_0k_0}l}(0).$$

We thus obtain (3.16). ∎

We conclude by Lemma 3.1 and equations (3.13), (3.17), (3.18) that

$$\mathrm{Span}_{\mathbb{C}}\{\rho_W(F,\overline{F}),L_1\rho_W(F,\overline{F}),\cdots,L_n\rho_W(F,\overline{F}),\{L_jL_k\rho_W(F,\overline{F})\}_{1\leq j,k\leq n}\}|_{Z=0} = \mathbb{C}^{N+1}.$$

This implies that F is 2−nondegenerate at 0 in the sense of Definition 2.6 ([La1]). By the results of [La1, La2], F is real-analytic (resp. smooth) near 0, as required. This establishes the part (1) of Theorem 1. In the case when M,M' are real algebraic, again as F is 2−nondegenerate at 0, then it follows from Theorem 5 in [La3] that F is algebraic. This establishes part (2) of Theorem 1. To provide some details to the readers on how $k-$nondegeneracy can be used to establish regularity of the map, we sketch a proof here for the algebraicity case.

For that we first recall some needed notations. Let $M\subset U(\subset\mathbb{C}^n)$ be a closed real-algebraic subset defined by a family of real polynomials $\{\rho_\alpha(Z,\overline{Z})\}$, where Z is the coordinates of $\mathbb{C}^n$. Note the complexification $\rho_\alpha(Z,W)$ of $\rho_\alpha(Z,\overline{Z})$ is complex algebraic over $U\times conj(U)$ with $conj(U):=\{W:\overline{W}\in U\}$ for each α. Then the complexification $\mathcal{M}$ of M is the complex-algebraic subset in

$U \times conj(U)$ defined by $\rho_\alpha(Z,W)=0$ for each α. Let $W \in \mathbb{C}^n$. The Segre variety of M associated with the point W is defined by $Q_W := \{Z : (Z,\overline{W}) \in \mathcal{M}\}$. Recall the following fact from [Hu1].

Proposition 3.2. *(Lemma 3,* [Hu1]*) Let M be a piece of strongly pseudoconvex real algebraic hypersurface. If g is a holomorphic function defined near $p \in M$ and algebraic on any Segre variety Q_z for $z \approx p$, then g is algebraic.*

The following result shows $k-$nondegeneracy of the map ensures the algebraicity (Note Lemma 3.3 is contained in Theorem 5 of [La3]).

Lemma 3.3. *Let $M \subset \mathbb{C}^{n+1}$ be a germ of real algebraic strongly pseudoconvex hypersurface, $M' \subset \mathbb{C}^{N+1}, N \geq n$, a real algebraic generic CR submanifold. Let F be a CR map from M to M' of class $C^k, k \geq 1$. Assume that F is $k-$nondegenerate. Then F is algebraic.*

Proof. First since F is $k-$nondegenerate, it follows from [La2] that F extends holomorphically to a neighborhood of M. We write

$$M := \{Z \in U : r(Z,\overline{Z})=0\}; \quad M' := \{W \in V : \rho_\mu(W,\overline{W})=0, 1 \leq \mu \leq d\}.$$

Here U,V are small open sets in $\mathbb{C}^{n+1}, \mathbb{C}^{N+1}$, respectively, and r, ρ_μ are real polynomials. Fix $p \in M$. Assume $\frac{\partial r}{\partial z_{n+1}}(p) \neq 0$. Write $L_j = \frac{\partial r}{\partial \overline{z_{n+1}}}\frac{\partial}{\partial \overline{z_j}} + \frac{\partial r}{\partial \overline{z_j}}\frac{\partial}{\partial \overline{z_{n+1}}},\ 1 \leq j \leq n$. Then $\{L_j\}_{1 \leq j \leq n}$ forms a basis of CR vector fields along M near p. Write $L^\alpha = L_1^{\alpha_1}\cdots L_n^{\alpha_n}$ for any multi-index $\alpha=(\alpha_1,...,\alpha_n)$. Since F is $k-$nondegenerate, we can find multi-indices $\alpha^1,...,\alpha^{N+1}$ with each $|\alpha^i| \leq k$ and $1 \leq \mu_1,...,\mu_{N+1} \leq d$, such that

$$\mathrm{Span}_{\mathbb{C}}\{L^{\alpha^i}\rho_{\mu_i,W}(F(Z),\overline{F(Z)})|_{Z=p} : 1 \leq i \leq N+1\} = \mathbb{C}^{N+1}. \tag{3.20}$$

We obtain by complexification that

$$\rho_\mu(F(Z),\overline{F(\xi)})=0,\ 1 \leq \mu \leq d, \tag{3.21}$$

for $r(Z,\overline{\xi})=0,\ (Z,\xi) \in U \times U$. Here U is a neighborhood of p in $\mathbb{C}^{n+1}$. Write $\xi=(\xi_1,...,\xi_n)$, and

$$\mathcal{L}_j = \frac{\partial r(Z,\xi)}{\partial \overline{\xi_{n+1}}}\frac{\partial}{\partial \overline{\xi_j}} - \frac{\partial r(Z,\xi)}{\partial \overline{\xi_j}}\frac{\partial}{\partial \overline{\xi_{n+1}}}, 1 \leq j \leq n.$$

They are vector fields along $\mathcal{M} := \{(Z,\xi) \in U \times U : r(Z,\xi)=0\}$. Similarly as L^α, we write $\mathcal{L}^\alpha = \mathcal{L}_1^{\alpha_1}...\mathcal{L}_n^{\alpha_n}$ for a multiindex $\alpha=(\alpha_1,...,\alpha_n)$. Applying $\mathcal{L}^{\alpha^i}, 1 \leq i \leq N+1$, to (3.21) with $\mu=\mu_i$, we get

$$\mathcal{L}^{\alpha^i}\rho_{\mu_i}(F(Z),\overline{F(\xi)})=0,\ 1 \leq i \leq N+1. \tag{3.22}$$

for any $(Z,\xi) \in U \times U$ with $r(Z,\overline{\xi})=0$. As $r(z,\xi), \rho_\mu(z,\xi)$ are polynomials, we conclude the coefficients of the operators $\mathcal{L}_j'$s are polynomials in (Z,ξ), and thus the left hand side of (3.22) is polynomial in the variables corresponding to $(Z,\xi,F(Z),(\mathcal{L}^\alpha\overline{F(\xi)})_{|\alpha| \leq k})$. We also note that for a function $h(\cdot,\cdot)$ holomorphic in $(Z,\overline{\xi}) \in U \times conj(U)$, we have $\mathcal{L}_j h(Z,\overline{\xi})|_{Z=\xi=p} = L_j h(Z,\overline{Z})|_{Z=p}$. As a consequence of (3.20), we have

$$\mathrm{Span}_{\mathbb{C}}\{\mathcal{L}^{\alpha^i}\rho_{\mu_i,W}(F(Z),\overline{F(\xi)}) : 1 \leq i \leq N+1\} = \mathbb{C}^{N+1}, \tag{3.23}$$

for any $(Z,\xi) \in O \times O$, where $O \subset U$ is a small neighborhood of p in $\mathbb{C}^{n+1}$. Fix $\xi = q \in O$. By (3.23), the $(N+1)$ equations in (3.22) give a vector-valued algebraic function $\mathbf{\Phi}(Z,W) = (\Phi_1, \cdots, \Phi_{N+1})$ in $\mathbb{C}^{n+1} \times \mathbb{C}^{N+1}$ such that the matrix $\mathbf{\Phi}_W$ is nondegenerate near $(\widetilde{p}, F(\widetilde{p}))$ for each $\widetilde{p} \in O$. Moreover, $\mathbf{\Phi}(Z, F(Z)) = 0$, for $Z \in Q_q := \{Z \in O : r(Z, \overline{q}) = 0\}$. We apply the algebraic version of implicit function theorem (cf. [Hu1]) to conclude that F is algebraic on Q_q for each $q \in O$. We thus obtain the algebraicity of F by Proposition 3.2.■ □

Remark 3.4. If $M' \subset \mathbb{C}^{N+1}$ is a real hypersurface as described in Theorem 1, then $N \leq n + n^2$ by the equation (3.13).

4. CR geometric properties of smooth boundaries of classical domains and applications

In this section, we will systematically investigate the smooth boundaries of classical domains of each type. As applications, we prove propositions 1.2-1.5. We first recall the notions and some basic properties of Hermitian symmetric spaces. Let X_0 be a Hermitian symmetric space of noncompact type and X its compact dual. They can be expressed as coset spaces of Lie groups:

$$X_0 = G_0/K, \;\; X = G/P,$$

where G_0 and G are the largest connected Lie groups of automorphisms of X_0 and X respectively, and the complex Lie subgroup G is the complexification of the real Lie group G_0. Here K is the isotropy subgroup of G_0, and P is a maximal parabolic subgroup of G. One can arrange in such a way that $K = G_0 \cap P$ by using the fact $G_0 \subset G$. This leads to a natural embedding (Borel embedding) β of X_0 into X as an open subset: $\beta(gK) = gP \in G/P = X$. In this way, every automorphism of X_0 extends to an automorphism of X, and X_0 becomes an open G_0-orbit in X.

In a more explicit way, the Harish-Chandra embedding realizes X_0 as a bounded domain in the holomorphic tangent space at some base point $x_0 \in X_0$. Roughly speaking, there is a complex Euclidean space $\mathfrak{m}^+ \subset X$, whose complement is a subvariety of lower dimension in X, such that $X_0 \subset \mathfrak{m}^+ \subset X$. The inclusion $X_0 \subset \mathfrak{m}^+$ is a canonical realization of X_0 as a bounded symmetric domains. By the boundary orbit theorem (See page 287 , [Wo]), the topological boundary of X_0 is stratified into several G_0-orbits in X. The smooth part of the boundary is one of these orbits (cf. Lemma 2.2.3 in [MN]). The readers are referred to [Wo] for more details on the fine structure theory of Hermitian symmetric spaces.

Recall irreducible bounded symmetric domains can be classified as Cartan's four types of classical domains and two exceptional cases (cf. [H], [M1]). We will discuss the smooth boundary separately for each type of classical domains, and will prove the smooth boundary of an irreducible classical domain with rank at least 2 must be uniformly 2−nondegenerate. As a byproduct, we compute the number of nonzero eigenvalues of the Levi form at a smooth boundary point. This was also obtained by Mok using a different approach (See Lemma 3, [M5]). As shown in [M5], the number of nonzero eigenvalues of the Levi form is closely related to the dimension of the VMRT of the compact dual of the classical domain (See Hwang-Mok [HM]).

4.1. **Smooth boundary of type I domain.** We first study the boundary of the type I domain $D^I_{p,q} = \{Z \in \mathbb{C}^{p\times q} : I_p - Z\overline{Z}^t > 0\}$. Recall its boundary is given by $\partial D^I_{p,q} = \{Z \in \mathbb{C}^{p\times q} : I_p - Z\overline{Z}^t \geq 0; \det(I_p - Z\overline{Z}^t) = 0\}$. To better illustrate the boundary, we recall the following basic fact from linear algebra. Let Z be a $p \times q (p \leq q)$ matrix. Then there exist a $p \times p$ unitary matrix U and a $q \times q$ unitary matrix V such that

$$Z = U \begin{pmatrix} r_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_p & 0 & \cdots & 0 \end{pmatrix} V$$

with $r_1 \geq r_2 \geq \cdots \geq r_p \geq 0$. The above equation is called the singular value decomposition of Z. The r_i's are called the singular values of Z. Their squares give the eigenvalues of $Z\overline{Z}^t$.

Although we will not use this in the proof, for the convenience of the readers, we mention the boundary of the type I domain $D^I_{p,q}$ is given by

$$\partial D^I_{p,q} = \{Z \in \mathbb{C}^{p\times q} : 1 = r_1 \geq r_2 \geq \cdots \geq r_p \geq 0\}$$

Here r_i's are the singular values of Z as above. The smooth part of $\partial D^I_{p,q}$ is given by (cf. [Wo], [M3] or Proposition 2.1, [LT]):

$$S_{p,q} = \{Z \in \mathbb{C}^{p\times q} : 1 = r_1 > r_2 \geq \cdots \geq r_p \geq 0\}.$$

In particular, $\mathrm{diag}(1, 0, \cdots, 0)$ is a smooth boundary point of $D^I_{p,q}$.

Write $\rho(Z, \overline{Z}^t) = \det(I_p - Z\overline{Z}^t)$. Then ρ is a local defining function of $\partial D^I_{p,q}$ near any smooth point (cf. Lemma 2 in [M5]). Let $G(p, q)$ be the compact dual of $D^I_{p,q}$, i.e., the Grassmannian space consisting of p planes in $\mathbb{C}^{p+q}$. By the boundary orbit theorem in [Wo], given any two smooth boundary points $Z, W \in S_{p,q} \subset \partial D^I_{p,q} \subset G(p, q)$, there is an automorphism $g \in G_0$ of $D^I_{p,q}$ that extends to some automorphism $\widetilde{g} \in G$ of $G(p, q)$ such that $\widetilde{g}(Z) = W$. This fact will simplify a lot our computation later.

Proposition 4.1. *Let Z_0 be a smooth point of the boundary $\partial D^I_{p,q}$ of $D^I_{p,q} (q \geq p \geq 2)$. Then $\partial D^I_{p,q}$ is uniformly $2-$nondegenerate at Z_0. Moreover, the Levi form of $\partial D^I_{p,q}$ at Z_0 has exactly $p + q - 2$ nonzero eigenvalues (and they are of the same sign).*

Proof. As was discussed above, since (the extension of) the automorphism group G_0 of $D^I_{p,q}$ acts transitively on $S_{p,q}$, we can assume $Z_0 = \mathrm{diag}(1, 0, \cdots, 0)$. We will need the following lemma from algebra (cf. [XY2]).

Lemma 4.2. *We denote by* $Z\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$ *the determinant of the submatrix of* Z *formed by its* $i_1^{th}, ..., i_k^{th}$ *rows and* $j_1^{th}, ..., j_k^{th}$ *columns, where* $i_1 < \cdots < i_k$ *and* $j_1 < \cdots < j_k$*. Then*

$$\det(I_p - Z\overline{Z^t}) = 1 + \sum_{k=1}^{p} (-1)^k \left(\sum_{1 \le i_1 < i_2 < ... < i_k \le p, 1 \le j_1 < j_2 < ... < j_k \le q} \left| Z\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix} \right|^2 \right). \tag{4.1}$$

Write the coordinates in the matrix form: $Z = (z_{ij})_{1 \le i \le p, 1 \le j \le q}$. We will also write Z as a row vector in $\mathbb{C}^{pq}$:

$$Z = (z_{11}, \cdots, z_{1q}, z_{21}, \cdots, z_{2q}, \cdots, z_{p1}, \cdots, z_{pq}).$$

Write $\widetilde{Z}$ as the components of Z with z_{11} dropped.

Write $\rho_Z = (\frac{\partial \rho}{\partial z_{kl}})_{1 \le k \le p, 1 \le l \le q}$. We arrange it to be $(pq)-$dimensional row vector and will say the component $\frac{\partial \rho}{\partial z_{kl}}$ is at the $(k,l)^{\text{th}}$ position. One can readily check with the help of Lemma 4.2 that

$$\rho_Z|_{Z_0} = \frac{\partial \rho}{\partial Z}|_{Z_0} = (-1, 0, \cdots, 0). \tag{4.2}$$

Here clearly the component "-1" is at the $(1,1)^{\text{th}}$ position. Let (i,j) be a pair of integers in $S = \{(i,j) : 1 \le i \le p, 1 \le j \le q, \ (i,j) \ne (1,1)\}$. Set L_{ij} to be the CR tangent vector fields along $\partial D^I_{p,q}$ near Z_0 as follows:

$$L_{ij} = -\frac{\partial \rho}{\partial \overline{z}_{11}} \frac{\partial}{\partial \overline{z}_{ij}} + \frac{\partial \rho}{\partial \overline{z}_{ij}} \frac{\partial}{\partial \overline{z}_{11}}. \tag{4.3}$$

It is readily checked by Lemma 4.2 that

$$L_{ij} = (z_{11} + O(||\widetilde{Z}||^2)) \frac{\partial}{\partial \overline{z}_{ij}} - (z_{ij} + H_{ij}(Z)\overline{z}_{11} + O(||\widetilde{Z}||^2)) \frac{\partial}{\partial \overline{z}_{11}}. \tag{4.4}$$

Here $H_{ij}(Z) = z_{11}z_{ij} - z_{1j}z_{i1}$ is holomorphic quadratic function in Z when $i \ne 1$ and $j \ne 1$. Otherwise, $H_{ij}(Z) \equiv 0$. Note we always have $H_{ij}(Z_0) = 0$.

Using the above equations, we calculate,

$$L_{ij}|_{Z_0} = \frac{\partial}{\partial \overline{z}_{ij}}, \ (i,j) \in S. \tag{4.5}$$

$$L_{st}L_{ij}|_{Z_0} = \frac{\partial}{\partial \overline{z}_{st}} \frac{\partial}{\partial \overline{z}_{ij}}, \ (i,j), (s,t) \in S. \tag{4.6}$$

Fix $(i,j) \in S$. Then for any $(k,l) \in S$, the $(k,l)^{\text{th}}$ component of $L_{ij}\rho_Z|_{Z_0}$ is given by

$$h(i,j,k,l) := \frac{\partial^2 \rho}{\partial \overline{z}_{ij} \partial z_{kl}}|_{Z_0}.$$

We use the explicit formula in Lemma 4.2 to compute that,

- If $(i,j) \ne (k,l)$, then $h(i,j,k,l) = 0$;
- If $(i,j) = (k,l) \in S$ with $i = k = 1$, then $h(1,j,1,j) = \frac{\partial^2}{\partial \overline{z}_{1j} \partial z_{1j}}(-|z_{1j}|^2) = -1$;
- If $(i,j) = (k,l) \in S$ with $j = l = 1$, then $h(i,1,i,1) = \frac{\partial^2}{\partial \overline{z}_{i1} \partial z_{i1}}(-|z_{i1}|^2) = -1$;

- If $(i,j)=(k,l)\in S$ with $i=k\geq 2, j=l\geq 2$, then

$$h(i,j,i,j)=\frac{\partial^2}{\partial\overline{z}_{ij}\partial z_{ij}}(-|z_{ij}|^2+|z_{11}z_{ij}-z_{1j}z_{i1}|^2)|_{Z_0}=0.$$

Thus we have if $(1,j)\in S$, i.e., $2\leq j\leq q$, then

$$L_{1j}\rho_Z|_{Z_0}=(0,\cdots,0,-1,0,\cdots,0). \tag{4.7}$$

Here the component "-1" is at the $(1,j)^{\text{th}}$ position. Similarly, if $(i,1)\in S$, i.e., $2\leq i\leq p$,

$$L_{i1}\rho_Z|_{Z_0}=(0,\cdots,0,-1,0,\cdots,0), \tag{4.8}$$

where the component "-1" is at the $(i,1)^{\text{th}}$ position. When $(i,j)\in S$ with $i\neq 1, j\neq 1$, we have

$$L_{ij}\rho_Z|_{Z_0}=(0,\cdots,0) \tag{4.9}$$

with all components equal zero. Moreover, for $(i,j)\in S$ with $i\neq 1, j\neq 1$,

$$L_{i1}L_{1j}\rho_Z|_{Z_0}=\frac{\partial^2}{\partial\overline{z}_{i1}\partial\overline{z}_{1j}}\rho_Z|_{Z_0}=(0,\cdots,0,-1,0,\cdots,0), \tag{4.10}$$

where the component "-1" is at the $(i,j)^{\text{th}}$ position. Indeed, for any $1\leq s\leq p, 1\leq t\leq q$, the $(s,t)^{\text{th}}$ component of $\frac{\partial^2}{\partial\overline{z}_{i1}\partial\overline{z}_{1j}}\rho_Z|_{Z_0}$ is given by

$$\frac{\partial^3\rho}{\partial\overline{z}_{i1}\partial\overline{z}_{1j}\partial z_{st}}|_{Z_0}.$$

Note every term in the expansion of ρ is annihilated by $\frac{\partial^3}{\partial\overline{z}_{i1}\partial\overline{z}_{1j}\partial z_{st}}$ when evaluated at Z_0 unless $(s,t)=(i,j)$. In the case when $(s,t)=(i,j)$, the only nonzero term is

$$\frac{\partial^3}{\partial\overline{z}_{i1}\partial\overline{z}_{1j}\partial z_{ij}}\left(\begin{vmatrix} z_{11} & z_{1j}\\ z_{i1} & z_{ij}\end{vmatrix}\begin{vmatrix} \overline{z_{11}} & \overline{z_{1j}}\\ \overline{z_{i1}} & \overline{z_{ij}}\end{vmatrix}\right)|_{Z_0}=-1.$$

This establishes the equation (4.10). It follows from equations (4.7-4.9) that,

$$\text{rank}\begin{pmatrix}\rho_Z\\ (L_{ij}\rho_Z)_{(i,j)\in S}\end{pmatrix}|_{Z_0}=p+q-1. \tag{4.11}$$

Here $(L_{ij}\rho_Z)_{(i,j)\in S}$ denotes the matrix with $|S|=pq-1$ rows, where in each row it is the vectors $L_{ij}\rho_Z$. This implies the Levi form of $\partial D^I_{p,q}$ at Z_0 has precisely $p+q-2$ nonzero eigenvalues. They are of the same sign as $D^I_{p,q}$ is pseudoconvex. One the other hand, equations (4.7), (4.8), (4.10) imply

$$\text{rank}\begin{pmatrix}\rho_Z\\ (L_{1j}\rho_Z)_{2\leq j\leq q}\\ (L_{i1}\rho_Z)_{2\leq i\leq p}\\ (L_{i1}L_{1j}\rho_Z)_{2\leq i\leq p,2\leq j\leq q}\end{pmatrix}|_{Z_0}=pq. \tag{4.12}$$

Here $(L_{i1}L_{1j}\rho_Z)_{2\leq i\leq p,2\leq j\leq q}$ denotes $(p-1)(q-1)$ rows of vectors $L_{i1}L_{1j}\rho_Z$ with $2\leq i\leq p, 2\leq j\leq q$. The rows $(L_{1j}\rho_Z)_{2\leq j\leq q}, (L_{i1}\rho_Z)_{2\leq i\leq p}$ are defined similarly. This implies $\partial D^I_{p,q}$ is $2-$nondegenerate at p. We have thus established Proposition 4.1. $\square$

Proof of Proposition 1.2: Proposition 1.2 follows from Theorem 1 and Proposition 4.1. ∎

4.2. **Smooth boundary of type II domain.** Recall that we denote the coordinates in $\mathbb{C}_{II}^{\frac{m(m-1)}{2}}$ by a skew-symmetric $m \times m$ matrix Z. Note the boundary of the type II domain D_m^{II} is given by

$$\partial D_m^{II} = \{Z \in \mathbb{C}_{II}^{\frac{m(m-1)}{2}} : 1 = r_1 = r_2 \geq r_3 \geq \cdots \geq r_m \geq 0\},$$

where r_i's are the singular values of Z. The defining function of the smooth boundary of D_m^{II} is slightly different as $\det(I_m - Z\overline{Z}^t)$ is a reducible polynomial when Z is skew-symmetric. More precisely, we have the following lemma (cf. [H], [PS]).

Lemma 4.3. *Let I_n be the $n \times n$ identity matrix, Z be an $n \times n$ skew-symmetric matrix. Then*

$$\det(I_n - Z\overline{Z}^t) = \left(1 + \sum_{1 \leq k \leq n, 2|k} (-1)^{\frac{k}{2}} \left(\sum_{1 \leq i_1 < \ldots < i_k \leq n} \left| Z \begin{pmatrix} i_1 & \ldots & i_k \\ i_1 & \ldots & i_k \end{pmatrix} \right| \right) \right)^2. \tag{4.13}$$

Here "$2|k$" means that k is divisible by 2.

Moreover, the determinant of a skew-symmetric matrix, as a polynomial in the matrix entries, is also a complete square.

Lemma 4.4. *Let $A = (a_{ij})$ be a $2n \times 2n, n \geq 1$, skew-symmetric matrix. Then*

$$\det(A) = (pf(A))^2.$$

Here $pf(A)$ is the Pfaffian of A. It is a homogeneous polynomial in the matrix entries of degree n (For explicit formula of $pf(A)$, cf. [XY2]). Note that the determinant of an $n \times n$ skew-symmetric matrix for n odd is always zero. The Pfaffian of an $n \times n$ skew-symmetric matrix for n odd is defined to be zero.

Example 4.5. Let $a, b, c, d, e, f \in \mathbb{C}$. Set A be the skew-symmetric matrix:

$$A = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}.$$

Then $pf(A) = af - be + cd$ and $\det(A) = (af - be + cd)^2$.

Set

$$\rho = 1 + \sum_{1 \leq k \leq n, 2|k} (-1)^{\frac{k}{2}} \left(\sum_{1 \leq i_1 < \ldots < i_k \leq n} \left| Z \begin{pmatrix} i_1 & \ldots & i_k \\ i_1 & \ldots & i_k \end{pmatrix} \right| \right). \tag{4.14}$$

Then ρ is a local defining function for ∂D_m^{II} at any smooth point (cf. Lemma 2 in [M5]). By Lemma 4.3, 4.4 and Example 4.5, we have

$$\begin{aligned}\rho &= 1 - \sum_{1\leq i<j\leq m} |z_{ij}|^2 + \sum_{1\leq i<j<k<l\leq m} \left| Z\begin{pmatrix} i & j & k & l \\ i & j & k & l \end{pmatrix}\right| + O(||\widetilde{Z}||^4). \\ &= 1 - \sum_{1\leq i<j\leq m} |z_{ij}|^2 + \sum_{1\leq i<j<k<l\leq m} |z_{ij}z_{kl} - z_{ik}z_{jl} + z_{il}z_{jk}|^2 + O(||\widetilde{Z}||^4)\end{aligned} \tag{4.15}$$

If we set Z_0 to be the $m\times m$ skew-symmetric matrix:

$$Z_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{4.16}$$

then Z_0 is a smooth boundary point of D_m^{II} (cf. [Wo]). Given any two smooth boundary points $Z, W \in S_m^{II} \subset \partial D_m^{II} \subset X$, by [Wo], there is an automorphism $g \in G_0$ of D_m^{II} that extends to some automorphism $\widetilde{g} \in G$ of its compact dual such that $\widetilde{g}(Z) = W$.

Proposition 4.6. *Let Z_0 be a smooth point of the boundary ∂D_m^{II} of $D_m^{II}(m \geq 4)$. Then ∂D_m^{II} is uniformly $2-$nondegenerate at Z_0. Moreover, the Levi form of ∂D_m^{II} at Z_0 has exactly $2m-4$ nonzero eigenvalues (and they are of the same sign).*

Proof. Again as (the extension of) the automorphism group G_0 of D_m^{II} acts transitively on the smooth boundary, we can assume Z_0 is as in (4.16). Recall we denote the coordinates in $\mathbb{C}_{II}^{\frac{m(m-1)}{2}}$ as Z which is a skew-symmetric matrix. We will also write Z as a row vector:

$$Z = (z_{ij})_{i<j} = (z_{12}, \cdots, z_{1m}, z_{23}, \cdots, z_{2m}, \cdots, z_{mm}).$$

We write the row vector $\widetilde{Z}$ as the components of Z with z_{12} dropped. Write $\rho_Z = (\frac{\partial\rho}{\partial z_{kl}})_{1\leq k<l\leq m}$. We arrange it to be $\frac{m(m-1)}{2}-$dimensional row vector and will say the component $\frac{\partial\rho}{\partial z_{kl}}$ is at the $(k,l)^{\text{th}}$ position. By (4.15), one can easily check that,

$$\rho_Z|_{Z_0} = (-1, 0, \cdots, 0). \tag{4.17}$$

Here clearly the component "-1" is at the $(1,2)^{\text{th}}$ position. Let (i,j) be a pair of integers in the set $S := \{(i,j) : 1 \leq i < j \leq m, (i,j) \neq (1,2)\}$. Set L_{ij} to be the CR tangent vector fields along ∂D_m^{II} near Z_0 as follows:

$$L_{ij} = -\frac{\partial\rho}{\partial\overline{z}_{12}}\frac{\partial}{\partial\overline{z}_{ij}} + \frac{\partial\rho}{\partial\overline{z}_{ij}}\frac{\partial}{\partial\overline{z}_{12}}.$$

By (4.15), one can easily check that

$$L_{ij} = (z_{12} + O(||\widetilde{Z}||^2))\frac{\partial}{\partial\overline{z}_{ij}} - (z_{ij} + H_{ij}(Z)\overline{z}_{12} + O(||\widetilde{Z}||^2))\frac{\partial}{\partial\overline{z}_{12}}.$$

Here $H_{ij}(Z) = z_{12}z_{ij} - z_{1i}z_{2j} + z_{1j}z_{2i}$ is holomorphic quadratic function in Z when $j > i > 2$. Otherwise, $H_{ij}(Z) \equiv 0$. Note in all cases, we have $H_{ij}(Z_0) = 0$. Furthermore, using the above equations, we calculate

$$L_{ij}|_{Z_0} = \frac{\partial}{\partial \overline{z}_{ij}},\ (i,j) \in S. \tag{4.18}$$

$$L_{st}L_{ij}|_{Z_0} = \frac{\partial}{\partial \overline{z}_{st}}\frac{\partial}{\partial \overline{z}_{ij}},\ (i,j),(s,t) \in S. \tag{4.19}$$

Fix $(i,j) \in S$. Then for any $(k,l) \in S$, the $(k,l)^{\text{th}}$ component of $L_{ij}\rho_Z|_{Z_0}$ is given by

$$h(i,j,k,l) := \frac{\partial^2 \rho}{\partial \overline{z}_{ij}\partial z_{kl}}|_{Z_0}.$$

We use the explicit formula in (4.15) to compute that,

- If $(i,j) \neq (k,l)$, then $h(i,j,k,l) = 0$;
- If $(i,j) = (k,l) \in S$ and $i = k = 1$, then $h(1,j,1,j) = \frac{\partial^2}{\partial z_{1j}\partial \overline{z}_{1j}}(-|z_{1j}|^2) = -1$;
- If $(i,j) = (k,l) \in S$ and $i = k = 2$, then $h(2,j,2,j) = \frac{\partial^2}{\partial z_{2j}\partial \overline{z}_{2j}}(-|z_{2j}|^2) = -1$;
- If $(i,j) = (k,l) \in S$ and $i = k \geq 3$, then
$$h(i,j,i,j) = \frac{\partial^2}{\partial z_{ij}\partial \overline{z}_{ij}}(-|z_{ij}|^2 + |z_{12}z_{ij} - z_{1i}z_{2j} + z_{1j}z_{2i}|^2) = 0.$$

Thus we have if $(1,j) \in S$, i.e., $3 \leq j \leq m$, then

$$L_{1j}\rho_Z|_{Z_0} = (0,\cdots,0,-1,0,\cdots,0). \tag{4.20}$$

Here the component "-1" is at the $(1,j)^{\text{th}}$ position. Similarly, if $(2,j) \in S$, i.e., $3 \leq j \leq m$,

$$L_{2j}\rho_Z|_{Z_0} = (0,\cdots,0,-1,0,\cdots,0), \tag{4.21}$$

where the component "-1" is at the $(2,j)^{\text{th}}$ position. When $(i,j) \in S$ with $i \geq 3$, we have

$$L_{ij}\rho_Z|_{Z_0} = (0,\cdots,0) \tag{4.22}$$

with all components equal zero. Moreover,

$$L_{1i}L_{2j}\rho_Z|_{Z_0} = \frac{\partial^2}{\partial \overline{z}_{1i}\partial \overline{z}_{2j}}\rho_Z|_{Z_0} = (0,\cdots,0,-1,0,\cdots,0), \tag{4.23}$$

where the component "-1" is at the $(i,j)^{\text{th}}$ position. To verify that, we note for any $1 \leq s < t \leq m$, the $(s,t)^{\text{th}}$ component of $\frac{\partial^2}{\partial \overline{z}_{1i}\partial \overline{z}_{2j}}\rho_Z|_{Z_0}$ is given by

$$\frac{\partial^3 \rho}{\partial \overline{z}_{1i}\partial \overline{z}_{2j}\partial z_{st}}|_{Z_0}.$$

Note every term in the expansion of ρ is annihilated by $\frac{\partial^3}{\partial \overline{z}_{1i}\partial \overline{z}_{2j}\partial z_{st}}$ when evaluated at Z_0 unless $(s,t) = (i,j)$. In the case when $(s,t) = (i,j)$, the only nonzero term is

$$\frac{\partial^3}{\partial \overline{z}_{1i}\partial \overline{z}_{2j}\partial z_{ij}}\left(|z_{12}z_{ij} - z_{1i}z_{2j} + z_{1j}z_{2i}|^2\right)|_{Z_0} = -1.$$

This establishes the equation (4.23). It follows from equations (4.20)-(4.22) that the Levi form of ∂D_m^{II} has exactly $2m-4$ nonzero eigenvalues. We conclude by (4.20)-(4.23) that

$$\operatorname{rank}\begin{pmatrix} \rho_Z \\ (L_{1i}\rho_Z)_{3\le i\le m} \\ (L_{2j}\rho_Z)_{3\le j\le m} \\ (L_{1i}L_{2j}\rho_Z)_{3\le i<j\le m}\end{pmatrix}\Big|_{Z_0} = \frac{m(m-1)}{2}. \tag{4.24}$$

This establishes Proposition 4.6. □

Proof of Proposition 1.3: Proposition 1.3 now follows from Theorem 1 and Proposition 4.6.■

4.3. **smooth boundary of the type III domain.** As above, denote the coordinates in $\mathbb{C}_{III}^{\frac{m(m+1)}{2}}$ by a symmetric matrix Z. We will write Z as a row vector

$$Z = (z_{ij})_{i\le j} = (z_{11}, \cdots, z_{1m}, z_{22}, \cdots, z_{2m}, \cdots, z_{mm}).$$

Note the boundary of the type III domain D_m^{III} is given by

$$\partial D_m^{III} = \{Z \in \mathbb{C}_{III}^{\frac{m(m+1)}{2}} : 1 = r_1 \ge r_2 \ge \cdots \ge r_m \ge 0\}.$$

Here r_i's are the singular values of Z. The smooth part of the boundary is given by

$$S_m = \{Z \in \mathbb{C}_{III}^{\frac{m(m+1)}{2}} : 1 = r_1 > r_2 \ge \cdots \ge r_m \ge 0\}.$$

In particular, $Z_0 = \mathrm{diag}(1, 0, \cdots, 0)$ is a smooth boundary point of D_m^{III}. Write $\rho(Z, \overline{Z}) = \det(I_m - Z\overline{Z}^t)$. Then ρ is a local defining function of D_m^{III} in $\mathbb{C}_{III}^{\frac{m(m+1)}{2}}$ at Z_0.

By [Wo], given any two smooth boundary points $Z, W \in S_m^{III} \subset \partial D_m^{III}$, there is an automorphism $g \in G_0$ of D_m^{III} that extends to some automorphism $\widetilde{g} \in G$ of its compact dual such that $\widetilde{g}(Z) = W$.

Proposition 4.7. *Let Z_0 a smooth point of the boundary ∂D_m^{III} of $D_m^{III}(m \ge 2)$. Then ∂D_m^{III} is uniformly $2-$nondegenerate at Z_0. Moreover, the Levi form of ∂D_m^{III} at Z_0 has exactly $m-1$ nonzero eigenvalues (and they are of the same sign).*

Proof. The proof of this case will be very similar to type I. As before, since (the extension of) the automorphism group G_0 of D_m^{III} acts transitively on S_m, we can assume $Z_0 = \mathrm{diag}(1, 0, \cdots, 0)$. Set $\widetilde{Z}$ as the components of Z with z_{11} dropped. We derive from Lemma 4.2 the following fact. Recall we write $Z\begin{pmatrix} i,j \\ k,l\end{pmatrix}$ to denote the determinant of the submatrix of Z formed the $i^{\text{th}}, j^{\text{th}}$ rows and the $k^{\text{th}}, l^{\text{th}}$ columns. In this notation, we always presume $i<j$ and $k<l$.

Lemma 4.8. *Let Z be a symmetric $m \times m$ matrix. Then*

$$\begin{aligned}\det\left(I_m - Z\overline{Z}^t\right) = 1 - \sum_{i=1}^{m} |z_{ii}|^2 - 2\sum_{1\le i<j\le m}|z_{ij}|^2 + \sum_{1\le i<j\le m}\left|Z\begin{pmatrix} i,j \\ i,j\end{pmatrix}\right|^2 \\ + 2\sum_{1\le i<k\le m, 1\le j\le l\le m}\left|Z\begin{pmatrix} i,j \\ k,l\end{pmatrix}\right|^2 + 2\sum_{1\le i<j<l\le m}\left|Z\begin{pmatrix} i,j \\ i,l\end{pmatrix}\right|^2 + O(||\widetilde{Z}||^4).\end{aligned} \tag{4.25}$$

We use the similar notion as above and write ρ_Z as a row vector:

$$\rho_Z = (\frac{\partial \rho}{\partial z_{kl}})_{1\le k\le l\le m}.$$

Using Lemma 4.8, one can check

$$\rho_Z|_{Z_0} = (-1, 0, \cdots, 0), \tag{4.26}$$

where the component "-1" is at the $(1,1)^{\text{th}}$ position. Let (i,j) be a pair of integers in the set $S := \{(i,j) : 1 \le i \le j \le m, (i,j) \ne (1,1)\}$. For $(i,j) \in S$, we defined L_{ij} as in (4.3). One can check the same equation as (4.4) holds for L_{ij} here with

$$H_{ij} = \begin{cases} (z_{11}z_{ii} - z_{1i}^2), \text{ if } i = j \ge 2; \\ (z_{11}z_{ij} - z_{1j}z_{i1} + z_{11}z_{ij} - z_{1i}z_{1j}), \text{ if } i \ne j, i \ge 2, j \ge 2; \\ 0, \text{ if } i = 1 \text{ or } j = 1. \end{cases}$$

In particular $H_{ij}(Z_0) = 0$ for all $(i,j) \in S$. Furthermore, we still have the identities in (4.5), (4.6). Now fix $(i,j) \in S$. For any $(k,l) \in S$, the $(k,l)^{\text{th}}$ component of $L_{ij}\rho_Z|_{Z_0}$ is given by

$$h(i,j,k,l) := \frac{\partial^2 \rho}{\partial \overline{z}_{ij}\partial z_{kl}}|_{Z_0}.$$

We use the explicit formula in Lemma 4.8 to derive the following lemma.

Lemma 4.9.
- *If $(i,j) \ne (k,l)$, then $h(i,j,k,l) = 0$;*
- *If $(i,j) = (k,l) \in S$, and $i = k = 1$, then $h(1,j,1,j) = \frac{\partial^2}{\partial \overline{z}_{1j}\partial z_{1j}}(-|z_{1j}|^2) = -1$;*
- *If $(i,j) = (k,l) \in S$ and $2 \le i = j$, then $h(i,i,i,i) = \frac{\partial^2}{\partial \overline{z}_{ii}\partial z_{ii}}(-|z_{ii}|^2 + |z_{11}z_{ii} - z_{1i}z_{1i}|^2)|_{Z_0} = 0$;*
- *If $(i,j) = (k,l) \in S$ and $2 \le i < j$, then*

$$h(i,j,i,j) = \frac{\partial^2}{\partial \overline{z}_{ij}\partial z_{ij}}(-2|z_{ij}|^2 + 2|z_{11}z_{ij} - z_{1j}z_{1i}|^2)|_{Z_0} = 0.$$

Thus we have if $(1,j) \in S$, i.e., $2 \le j \le m$, then

$$L_{1j}\rho_Z|_{Z_0} = (0, \cdots, 0, -1, 0, \cdots, 0). \tag{4.27}$$

Here the component "-1" is at the $(1,j)^{\text{th}}$ position. When $(i,j) \in S$ (i.e., $j \ge 2$), we have

$$L_{ij}\rho_Z|_{Z_0} = (0, \cdots, 0). \tag{4.28}$$

with all components equal zero. Moreover, for $(j,j) \in S$ with $j \geq 2$,

$$L_{1j}L_{1j}\rho_Z|_{Z_0} = \frac{\partial^2}{\partial \overline{z}_{1j}\partial \overline{z}_{1j}}\rho_Z|_{Z_0} = (0,\cdots,0,-1,0,\cdots,0). \tag{4.29}$$

where the component "-1" is at the $(j,j)^{\text{th}}$ position. Indeed, for any $1 \leq s \leq p, 1 \leq t \leq q$, the $(s,t)^{\text{th}}$ component of $\frac{\partial^2}{\partial \overline{z}_{1j}\partial \overline{z}_{1j}}\rho_Z|_{Z_0}$ is given by

$$\frac{\partial^3 \rho}{\partial \overline{z}_{1j}\partial \overline{z}_{1j}\partial z_{st}}|_{Z_0}.$$

Note every term in the expansion of ρ is annihilated by $\frac{\partial^3}{\partial \overline{z}_{1j}\partial \overline{z}_{1j}\partial z_{st}}$ when evaluated at Z_0 unless $(s,t) = (j,j)$. In the case when $(s,t) = (j,j)$, the only nonzero term is

$$\frac{\partial^3}{\partial \overline{z}_{1j}\partial \overline{z}_{1j}\partial z_{jj}}\left(\begin{vmatrix} z_{11} & z_{1j} \\ z_{1j} & z_{jj}\end{vmatrix}\begin{vmatrix} \overline{z_{11}} & \overline{z_{1j}} \\ \overline{z_{1j}} & \overline{z_{jj}}\end{vmatrix}\right)|_{Z_0} = -1.$$

This establishes the equation (4.29). Similarly, for $(i,j) \in S$ with $2 \leq i < j$,

$$L_{1i}L_{1j}\rho_Z|_{Z_0} = \frac{\partial^2}{\partial \overline{z}_{1i}\partial \overline{z}_{1j}}\rho_Z|_{Z_0} = (0,\cdots,0,-2,0,\cdots,0). \tag{4.30}$$

Here the component "-2" is at the $(i,j)^{\text{th}}$ position. Indeed, for any $1 \leq s \leq p, 1 \leq t \leq q$, the $(st)^{\text{th}}$ component of $\frac{\partial^2}{\partial \overline{z}_{1j}\partial \overline{z}_{1j}}\rho_Z|_{Z_0}$ is given by

$$\frac{\partial^3 \rho}{\partial \overline{z}_{1i}\partial \overline{z}_{1j}\partial z_{st}}|_{Z_0}.$$

Note every term in the expansion of ρ is annihilated by $\frac{\partial^3}{\partial \overline{z}_{1i}\partial \overline{z}_{1j}\partial z_{st}}$ when evaluated at Z_0 unless $(s,t) = (i,j)$. In the case when $(s,t) = (i,j)$, the only nonzero term is

$$\frac{\partial^3}{\partial \overline{z}_{1i}\partial \overline{z}_{1j}\partial z_{ij}}\left(\begin{vmatrix} z_{11} & z_{1j} \\ z_{1i} & z_{ij}\end{vmatrix}\begin{vmatrix} \overline{z_{11}} & \overline{z_{1j}} \\ \overline{z_{1i}} & \overline{z_{ij}}\end{vmatrix} + \begin{vmatrix} z_{11} & z_{1i} \\ z_{1j} & z_{ij}\end{vmatrix}\begin{vmatrix} \overline{z_{11}} & \overline{z_{1i}} \\ \overline{z_{1j}} & \overline{z_{ij}}\end{vmatrix}\right)|_{Z_0} = -2.$$

This establishes the equation (4.30). It follows from (4.27), (4.28) that

$$\text{rank}\begin{pmatrix} \rho_Z \\ (L_{ij}\rho_Z)_{(i,j)\in S}\end{pmatrix}|_{Z_0} = m.$$

Thus the Levi form of ∂D_m^{III} at p has exactly $m-1$ eigenvalues. The equations (4.27)-(4.30) imply

$$\text{rank}\begin{pmatrix} \rho_Z \\ (L_{1j}\rho_Z)_{2\leq j\leq m} \\ (L_{1i}L_{1j}\rho_Z)_{2\leq i\leq j\leq m}\end{pmatrix}|_{Z_0} = \frac{m(m+1)}{2}.$$

This shows ∂D_m^{III} is $2-$nondegenerate at p. We thus have established Proposition 4.7. □

Proof of Proposition 1.4: Proposition 1.4 follows from Theorem 1 and Proposition 4.7. ■

4.4. **Smooth boundary of the type IV domain.** Recall the type IV domain in $\mathbb{C}^m, m \geq 2$, is defined by

$$\{Z = (z_1, \cdots, z_m) \in \mathbb{C}^m : 1 - ||Z||^2 + \frac{1}{4}|ZZ^t|^2 > 0,\ ||Z||^2 < 2\}.$$

Here Z^t denotes the transpose of Z. Note(cf. [XY]) the smooth part of the boundary of the type IV domain is given by $\{Z = (z_1, \cdots, z_m) \in \mathbb{C}^m : 1 - ||Z||^2 + \frac{1}{4}|ZZ^t|^2 = 0, ||Z||^2 < 2\}$. The singular part is given by $\{Z = (z_1, \cdots, z_m) \in \mathbb{C}^m : 1 - ||Z||^2 + \frac{1}{4}|ZZ^t|^2 = 0, ||Z||^2 = 2\}$.

Proposition 4.10. *Let p be a smooth point of the boundary ∂D_m^{IV} of D_m^{IV}. Then ∂D_m^{IV} is uniformly $2-$nondegenerate at p. Moreover, the Levi form of ∂D_m^{IV} at p has exactly $m-2$ nonzero eigenvalues (and they are of the same sign).*

Proof. As the defining function in this case is much simpler than other types, we will provide a proof without using the boundary orbit theorem in [Wo]. Fix a smooth point p of ∂D_m^{IV}. Write $\mathrm{Re}(v)$ and $\mathrm{Im}(v)$ as the real and imaginary part for a vector v. Note there exists $\theta \in [0, 2\pi)$ such that $\mathrm{Re}(e^{i\theta}p)$ and $\mathrm{Im}(e^{i\theta}p)$ are orthogonal. Then there exists an orthogonal matrix T such that $e^{i\theta}pT = (a, bi, 0, \cdots, 0)$ with $a, b \in \mathbb{R}$. Note $e^{i\theta}T$ is an automorphism of ∂D_m^{IV}. By applying this automorphism, we can assume $p = (a, bi, 0, ..., 0)$. Since p is a smooth point on ∂D_m^{IV}, we have

$$1 - (a^2 + b^2) + \frac{1}{4}(a^2 - b^2)^2 = 0,\ a^2 + b^2 < 2. \tag{4.31}$$

Consequently, we must have $a \neq 0, b \neq 0$, and $a^2 - b^2 < 2$. Write $\rho = ||Z||^2 - \frac{1}{4}|ZZ^t|^2 - 1$. Then ∂D_m^{IV} is locally defined by $\rho = 0$ near p. Note $\frac{\partial \rho}{\partial z_1}|_p = a(1 - \frac{1}{2}(a^2 - b^2)) \neq 0$. We find a basis $\{L_2, \cdots, L_m\}$ for the CR vector fields near p along ∂D_m^{IV} : For each $2 \leq j \leq m$,

$$L_j = \frac{\partial \rho}{\partial \overline{z}_j}\frac{\partial}{\partial \overline{z}_1} - \frac{\partial \rho}{\partial \overline{z}_1}\frac{\partial}{\partial \overline{z}_j} = (z_j - \frac{1}{2}(z_1^2 + \cdots + z_m^2)\overline{z}_j)\frac{\partial}{\partial \overline{z}_1} - (z_1 - \frac{1}{2}(z_1^2 + \cdots + z_m^2)\overline{z}_1)\frac{\partial}{\partial \overline{z}_j};$$

Moreover,

$$\rho_Z = (\overline{z}_1 - \frac{1}{2}(\overline{z}_1^2 + \cdots + \overline{z}_m^2)z_1, \overline{z}_2 - \frac{1}{2}(\overline{z}_1^2 + \cdots + \overline{z}_m^2)z_2, \cdots, (\overline{z}_m - \frac{1}{2}(\overline{z}_1^2 + \cdots + \overline{z}_m^2)z_m).$$

We will write $S = ZZ^t = \sum_{j=1}^m z_j^2$. We now compute $L_2\rho_Z(Z)$, if we write $L_2\rho_Z(Z) = (A_1, \cdots, A_m)$, then

$$A_1 = z_2(1 - |z_1|^2) + z_1^2\overline{z}_2 - \frac{1}{2}S\overline{z}_2; A_2 = -\overline{z}_1 z_2^2 + \frac{1}{2}S\overline{z}_1 - z_1(1 - |z_2|^2).$$

For $k \geq 3, A_k = -\overline{z}_1 z_2 z_k + z_1\overline{z}_2 z_k$. Next if we write $L_3\rho_Z(Z) = (C_1, \cdots, C_m)$, then

$$C_1 = (z_3 - \frac{1}{2}S\overline{z}_3)(1 - |z_1|^2) + (z_1 - \frac{1}{2}S\overline{z}_1)(z_1\overline{z}_3); C_2 = -(z_3 - \frac{1}{2}S\overline{z}_3)(\overline{z}_1 z_2) + (z_1 - \frac{1}{2}S\overline{z}_1)(z_2\overline{z}_3).$$

$$C_3 = -\overline{z}_1 z_3^2 - z_1(1 - |z_3|^2) + \frac{1}{2}S\overline{z}_1.$$

For $k \geq 4, C_k = -\overline{z}_1 z_3 z_k + z_1 \overline{z}_3 z_k$. One can compute other $L_j \rho$ similarly. We evaluate them at p to get,

$$\rho_Z(p) = \left(a(1 - \frac{1}{2}(a^2 - b^2)), -bi(1 + \frac{1}{2}(a^2 - b^2)), 0, \cdots, 0 \right); \tag{4.32}$$

$$L_2 \rho_Z | p = \left(-bi(-1 + \frac{3}{2}a^2 + \frac{1}{2}b^2), -a(1 - \frac{1}{2}a^2 - \frac{3}{2}b^2), \cdots \right). \tag{4.33}$$

$$L_j \rho_Z|_p = (0, \cdots, 0, c_0, 0, \cdots, 0) \text{ with } c_0 = -a(1 - \frac{1}{2}(a^2 - b^2)) \neq 0, \quad 3 \leq j \leq m. \tag{4.34}$$

Here the component "c_0" is at the j^{th} position. We compute furthermore,

$$L_3^2 \rho_Z(p) = (D_1, \cdots, D_m),$$

where $D_1 = \frac{1}{2} c_0 (a^2 + b^2); D_2 = c_0 abi$. It follows from the above calculation that

$$\begin{aligned} \begin{vmatrix} \rho_Z(p) \\ L_2 \rho_Z(p) \\ \cdots \\ L_m \rho_Z(p) \end{vmatrix} &= c_0^{m-2} \begin{vmatrix} a(1 - \frac{1}{2}a^2 + \frac{1}{2}b^2) & -bi(1 + \frac{1}{2}a^2 - \frac{1}{2}b^2) \\ bi(1 - \frac{3}{2}a^2 - \frac{1}{2}b^2) & -a(1 - \frac{1}{2}a^2 - \frac{3}{2}b^2) \end{vmatrix} \\ &= c_0^{m-2} (a^2 + b^2) \left((a^2 + b^2) - 1 - \frac{1}{4}(a^2 - b^2)^2 \right) = 0. \end{aligned} \tag{4.35}$$

This together with (4.32), (4.33), (4.34) imply

$$\text{rank} \begin{pmatrix} \rho_Z(p) \\ L_2 \rho_Z(p) \\ \cdots \\ L_m \rho_Z(p) \end{pmatrix} = m - 1$$

Hence ∂D_m^{IV} is Levi-degenerate at p. Moreover, the Levi form at p has $m-2$ nonzero eigenvalues. On the other hand,

$$\begin{aligned} \begin{vmatrix} \rho_Z(p) \\ L_3 \rho_Z(p) \\ \cdots \\ L_m \rho_Z(p) \\ L_3^2 \rho_Z(p) \end{vmatrix} &= \pm\, c_0^{m-2} \begin{vmatrix} a(1 - \frac{1}{2}a^2 + \frac{1}{2}b^2) & -bi(1 + \frac{1}{2}a^2 - \frac{1}{2}b^2) \\ \frac{1}{2} c_0 (a^2 + b^2) & c_0 abi \end{vmatrix} = \pm c_0^{m-1} bi(\frac{3}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{4}(a^2 - b^2)^2) \\ &= \pm\, c_0^{m-1} bi(1 + \frac{1}{2}a^2 - \frac{1}{2}b^2). \end{aligned} \tag{4.36}$$

It is nonzero as $a^2 + b^2 < 2$. Hence ∂D_m^{IV} is $2-$nondegenerate at p. Proposition 4.10 is established. $\square$

Now part (1) of Proposition 1.5 follows from Theorem 1 and Proposition 4.10. We will postpone the proof of part (2) to Section 5.

5. Proof of Theorem 2 and 3

The following transversality result (Proposition 5.1) is folklore, and is sometimes referred to [Fo]. For completeness, we provide a short proof. It will be necessary for the proofs of Theorems 2 and 3.

Proposition 5.1. *Let $M \subset \mathbb{C}^n, M' \subset \mathbb{C}^N$ be smooth real hypersurfaces and $\Omega_1 \subset \mathbb{C}^n, \Omega_2 \subset \mathbb{C}^N$ open sets with M, M' as a part of their boundaries, respectively. Assume F is a holomorphic map from Ω_1 to Ω_2 with C^1 extension up to M, and maps M to M'. If Ω_2 is convex, then F is CR transversal along M.*

Proof. We first recall the following well-known fact (cf. [Ho], [Kr]) about convex sets.

Lemma 5.2. *If M' is a smooth piece of the boundary of a convex open set $D \subset \mathbb{R}^m$ and $b \in M'$, then there exists a neighborhood U of b and a smooth defining function ρ of D in U such that ρ is convex in U. That is,*

$$\sum_{j,k=1}^{m} \frac{\partial^2 \rho(x)}{\partial x_j \partial x_k} \xi_j \xi_k \geq 0 \tag{5.1}$$

for every $x \in U$ and $\xi = (\xi_1, \cdots, \xi_m) \in \mathbb{R}^m$.

Fix $a \in M$ and write $b = F(a)$. Lemma 5.2 yields that there is a smooth local defining function ρ of Ω_2 in some neighborhood U of b such that ρ is convex in U (in the sense of equation (5.1)). This implies in particular ρ is plurisubharmonic in U. Shrinking Ω_1 if necessary, we assume F maps Ω_1 to U. As F is holomorphic in Ω_1, we conclude $\rho \circ F$ is plurisubharmonic in Ω_1. The implies $\rho \circ F$ is in particular subharmonic in $2n$ real variables in Ω_1. Note $\rho \circ F < 0$ on Ω_1 and $\rho \circ F(a) = 0$ at every $a \in M$. It follows from the Hopf lemma for subharmonic functions that $\frac{\partial(\rho \circ F)}{\partial \nu}|_a > 0$. Here ν is the outward pointing normal unit vector of Ω_1 at $a \in M$. Hence F is transversal at $a \in M$. $\square$

We are now at the position to prove Theorem 2 and 3.

5.1. **Mappings into type I domains: Proof of Theorem 2.** By the assumption of Theorem 2, F extends C^2 smoothly up to an open piece M of $\partial\Omega$. We claim that F must send a dense open subset M_0 of M to the smooth part of $\partial D^I_{2,q}$. Suppose not, then F sends an open subset M^* of M to the singular part of $\partial D^I_{2,q}$, which is given by

$$\{Z \in \mathbb{C}^{2\times q} : Z\overline{Z}^t = I_2\}. \tag{5.2}$$

Write F in the matrix form $F = (F_{ij})_{1 \leq i \leq 2, 1 \leq j \leq q}$. Then it follows that $\sum_{j=1}^q |F_{1j}|^2 = \sum_{j=1}^q |F_{2j}|^2 = 1$ on M^*. Hence $F_1 = (F_{11}, \cdots, F_{1q})$ and $F_2 = (F_{21}, \cdots, F_{2q})$ maps M^* to the unit sphere $\partial\mathbb{B}^q$, which is of lower dimension than M^*. Since $\partial\Omega$ is of D'Angelo's finite type (See [DA]), and in particular, is strongly pseudoconvex at generic points. By shrinking M^* if necessary, we can assume M^* is strongly pseudoconvex. Then by Theorem 5.1 in [BX1], F_1 and F_2, and thus F must be constant on M^*, which further implies F is constant in Ω (cf. [BER2]). This is a contradiction.

Hence F must map a dense open subset M_0 to M to the smooth part of $\partial D^I_{2,q}$. As $D^I_{2,q}$ is convex, by Proposition 5.1, F is CR transversal along M_0. Then it follows from Proposition 1.2 that F is algebraic. ■

Proof of Corollary 1.8: We will show only for the case $n = p+2$ and the other cases are similar. We prove by contradiction. Suppose F has C^2 smooth extension up to an open piece M_0 of $\partial\Omega$. We claim that F maps an open dense subset of M_0 to the smooth part of $\partial D^I_{2,q}$. Indeed, as $\partial\Omega$ is of finite type, a generic boundary point is strongly pseudoconvex. By the same argument as above, we can use Theorem 5.1 in [BX1] to show that F cannot map any open subset of M_0 to the singular part of $\partial D^I_{2,q}$. Then by Proposition 5.1, F is transversal along M_0. Note the Levi form of $M_0 \subset \mathbb{C}^{q+2}$ at a generic point has $q+1$ nonzero eigenvalues. By Proposition 4.1, however, the Levi form of $\partial D^I_{2,q}$ has only q nonzero eigenvalues at a smooth point. This is a contradiction. ■

5.2. **Mappings into the Lie ball: Proof of Theorem 3.** Recall the tube domain τ^+_{m+1} in $\mathbb{C}^{m+1}$ over the future cone is given by

$$\tau^+_{m+1} := \{W = (w_1, \cdots, w_m, w_{m+1}) \in \mathbb{C}^{m+1} : (\mathrm{Im} w_{m+1})^2 > (\mathrm{Im} w_1)^2 + \cdots + (\mathrm{Im} w_m)^2, \mathrm{Im} w_{m+1} > 0\}.$$

Note the boundary of the tube domain over the future cone is given by $\partial\tau^+_{m+1} := \{W = (w_1, \cdots, w_m, w_{m+1}) \in \mathbb{C}^{m+1} : (\mathrm{Im} w_{m+1})^2 = (\mathrm{Im} w_1)^2 + \cdots + (\mathrm{Im} w_m)^2, \mathrm{Im} w_{m+1} \geq 0\}$. Write $\mathbb{T}^{m+1}$ as the smooth part of $\partial\tau^+_{m+1}$.

Recall the Lie ball (or the type IV classical domain) in $\mathbb{C}^{m+1}$ is given by $D^{IV}_{m+1} := \{z = (z_1, \cdots, z_m, z_{m+1}) \in \mathbb{C}^{m+1} : 1 - \sum_{j=1}^{m+1} |z_j|^2 + \frac{1}{4}|\sum_{j=1}^{m+1} z_j^2|^2 > 0, \sum_{i=1}^{m+1} |z_j|^2 < 2\}$. Before we proceed to prove Theorem 3, we recall the well-known fact that the future tube τ^+_{m+1} is biholomorphic to the Lie ball D^{IV}_{m+1}. More precisely, define the map $H : \tau^+_{m+1} \to D^{IV}_{m+1}$ (cf. [SV]) by

$$z_1 = 2\sqrt{2}i\frac{w_1}{(W+\mathbf{i})^2}, \cdots, z_m = 2\sqrt{2}i\frac{w_m}{(W+\mathbf{i})^2}, z_{m+1} = \sqrt{2}i\frac{1+W^2}{(W+\mathbf{i})^2},$$

where the $(m+1)-$dimensional vector $\mathbf{i} = (0, \cdots, 0, i)$. Here for any $W \in \mathbb{C}^{m+1}$, we write $W^2 := w_{m+1}^2 - w_1^2 - \cdots - w_m^2$. Then H gives a biholomorphic map from τ^+_{m+1} to D^{IV}_{m+1}.

The following lemma and its proof are inspired by a very interesting recent work of Mir [Mi] (One may alternatively use ideas from [LM] to prove Lemma 5.3 as well).

Lemma 5.3. *Let $M \subset \mathbb{C}^{n+1} (m > n \geq 1)$ be a strictly pseudoconvex real algebraic real hypersurface, F a CR transversal CR map of class C^{m-n+1} from M to $\mathbb{T}^{m+1}$. Then F extends to an algebraic holomorphic map.*

Proof. Fix $p_0 \in M$ and write $q_0 = F(p_0)$. Apply appropriate changes of coordinates in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{m+1}$, such that $p_0 = 0, q_0 = 0$ and the normalization in Proposition 2.7 holds. In particular, the target hypersurface is locally defined at 0 by (2.2). Thus we have the following equation holds on (an open piece of) M.

$$\rho(F(z),\overline{F(z)})=-\mathrm{Im}F_{m+1}(z)+\sum_{j=1}^{m}|F_j(z)|^2+\phi(F(z),\overline{F(z)}),\ \phi(Z,\overline{Z})=O(|Z|^3). \tag{5.3}$$

We apply a basis of CR vector fields $L_1,\cdots,L_n$ of M to this equation (5.3); together with (5.3) we obtain a system Φ of (n+1) equations. Note Φ is polynomial in $F,\overline{F}$ and $L\overline{F}$, and by Proposition 2.7, Φ is nondegenerate in $(F_1,\cdots,F_n,F_{m+1})$. We use the algebraic version of the implicit function theorem to this equation system to obtain a holomorphic algebraic map $\Psi(W,\Lambda,\Xi)$ valued in $\mathbb{C}^{n+1}$, defined in a neighborhood of $(0,\overline{L}F(0),0)\in\mathbb{C}^{m+1}\times\mathbb{C}^{n(m+1)}\times\mathbb{C}^{m-n}$, such that

$$\begin{aligned}(F_1,\cdots,F_n,F_{m+1})&=\Psi(\overline{F},L\overline{F},F_{n+1},\cdots,F_m)\\&=(\Psi_1(\overline{F},L\overline{F},F_{n+1},\cdots,F_m),\cdots,\Psi_{n+1}(\overline{F},L\overline{F},F_{n+1},\cdots,F_m)).\end{aligned} \tag{5.4}$$

To establish the algebraicity, we will first need to show F is real-analytic on some open piece of M, and thus extends holomorphically to a neighborhood in $\mathbb{C}^{n+1}$ of this open piece. But we will postpone the proof of this real-analyticity, and at this point, first assume F is real-analytic on some open piece of M. We still write this open piece as M.

We borrow ideas from [Mi] and make the following claim.

Claim 1: Let $l\le m-n$. Assume the map F splits as $F=(\tilde{F},\hat{F})\in\mathbb{C}^r\times\mathbb{C}^{m+1-r}$ for some $1\le r\le m-n$ and satisfies in a neighborhood $M_0\subset M$ of some $p\in M_0$:

$$\hat{F}=\Theta(z,\overline{z},(L^\alpha\overline{F})_{|\alpha|\le l},\tilde{F}(z)), \tag{5.5}$$

for some $\mathbb{C}^{m-r+1}-$valued complex algebraic function Θ defined in a neighborhood of $(p,\overline{p},((L^\alpha\overline{F})(p))_{|\alpha|\le l},\tilde{F}(p))$. Then we can pick one component of $\tilde{F}$, denoted by $\tilde{F}_1$, such that the following holds in a neighborhood of some point $q\in M_0$:

$$(\hat{F},\tilde{F}_1)=\eta(z,\overline{z},(L^\alpha\overline{F})_{|\alpha|\le l+1},\tilde{F}_2).$$

Here $\tilde{F}=(\tilde{F}_1,\tilde{F}_2)$. And η is a $\mathbb{C}^{m-r+2}-$valued algebraic map in a neighborhood of $(q,\overline{q},((L^\alpha\overline{F})(q))_{|\alpha|\le l+1},\tilde{F}_2(q))$.

Proof of Claim 1: Here we can use a similar argument as in [Mi] (Lemma 2.1 on page 7) except that we will need to take care of the algebraicity. To do that we use the ideas from [Hu1]. Write $w=(\tilde{w},\hat{w})\in\mathbb{C}^r\times\mathbb{C}^{m+1-r}$ for the coordinates in $\mathbb{C}^{m+1}$ associated with the splitting $F=(\tilde{F},\hat{F})$. We differentiate (5.5) to get

$$0=L_j(\Theta(z,\overline{z},((L^\alpha\overline{F})(z))_{|\alpha|\le l},\tilde{F}(z))=:\Phi_j(z,\overline{z},((L^\alpha\overline{F})(z))_{|\alpha|\le l+1},\tilde{F}(z)),1\le j\le n. \tag{5.6}$$

As M is real algebraic, we can assume L_j has polynomial coefficients in z. Using the fact that the derivative of an algebraic function is still algebraic (See [Hu1]), we conclude each $\Phi_j=\Phi_j(z,\xi,(\Lambda_\alpha)_{|\alpha|\le l+1},\tilde{w})$ is a $\mathbb{C}^{m+1-r}-$valued algebraic function in a neighborhood of $(p,\overline{p},((L^\alpha\overline{F})(p))_{|\alpha|\le l+1},\tilde{F}(p))$. We will proceed in two cases.

Case 1: There is some $1\le j_0\le n$ such that $\Phi_j(z,\overline{z},((L^\alpha\overline{F})(z))_{|\alpha|\le l+1},\tilde{w})\not\equiv 0$ for $(z,\tilde{w})$ in some open piece $M_1\times V\subset M_0\times\mathbb{C}^r$ containing $(p,\tilde{F}(p))$. Then there exists a multi-index $\beta_0\in\mathbb{N}^r$

with the minimal norm $|\beta_0|$ such that $(\Phi_j)_{\tilde{w}^{\beta_0}}(z,\overline{z},((L^\alpha\overline{F})(z))_{|\alpha|\le l+1},\tilde{F}(z)) \not\equiv 0$ along M_1. And $(\Phi_j)_{\tilde{w}^{\beta}}(z,\overline{z},((L^\alpha\overline{F})(z))_{|\alpha|\le l+1},\tilde{F}(z)) \equiv 0$ along M_1 if $|\beta| < |\beta_0|$. The conclusion then follows from the algebraic version of the implicit function theorem.

Case 2: For every $1 \le j \le n$, $\Phi_j(z,\overline{z},((L^\alpha\overline{F})(z))_{|\alpha|\le l+1},\tilde{w}) \equiv 0$ for $(z,\tilde{w})$ in some open piece $M_1 \times V \subset M \times \mathbb{C}^r$ containing $(p,\tilde{F}(p))$. By (5.6), $\Theta(z,\overline{z},((L^\alpha\overline{F})(z))_{|\alpha|\le l},\tilde{w})$ is CR along $M_1\times V$. As $M_1\times V$ is minimal, by the standard edge-of-the-wedge theorem, we conclude that $\Theta(z,\overline{z},((L^\alpha\overline{F})(z))_{|\alpha|\le l},\tilde{w})$ extends to a holomorphic function, denoted by $A(z,\tilde{w})$ defined in a neighborhood $U\times V$ of $M_1\times V$ in $\mathbb{C}^N\times\mathbb{C}^r$. We claim A is algebraic in $(z,\tilde{w})$. It suffices to show A is algebraic in z and $\tilde{w}$ separately (cf. [Hu1]). From the algebraicity of Θ, it is clear that A is algebraic in $\tilde{w}$. To show the algebraicity of A in z, we fix $\tilde{w}_0 \in V$ and set $\psi(z) = A(z,\tilde{w}_0)$. Then $\psi(z)$ is holomorphic in U and

$$\psi(z) = \Theta(z,\overline{z},((L^\alpha\overline{F})(z))_{|\alpha|\le l},\tilde{w}_0) \text{ for } z \in M_1.$$

Since F extends holomorphically to a neighborhood of M, we can complexify the above equation. That is, we replace $\overline{z}$ by the conjugate $\overline{\xi}$ of an independent variable ξ. Then fix $\xi = \xi_0 \approx p_0 \in M_1$ to conclude $\psi(z)$ is algebraic when restricted to Q_{ξ_0}. As ξ_0 is arbitrary, we apply Proposition 3.2 to get the algebraicity of ψ and the algebraicity of A in z follows as well.

Note by (5.5), we have

$$\hat{F}(z) = A(z,\tilde{F}(z)),\ z \in M_1. \tag{5.7}$$

We next claim that

$$\rho(\tilde{w},A(z,\tilde{w}),\overline{\tilde{w}},\overline{A(z,\tilde{w})}) \not\equiv 0, (z,\tilde{w}) \in M_1\times V. \tag{5.8}$$

Suppose not, i.e., the left hand side of the above equation vanishes everywhere along $M_1\times V$. If we set $\Psi_t(z) = (t+\tilde{F}(z), A(z,t+\tilde{F}(z)))$, where $t \in \mathbb{C}^r$ sufficiently close to 0 and $z \in M_1$ close to p, then $\rho(\Psi_t(z),\overline{\Psi_t(z)}) \equiv 0$. This implies $(\Psi_t)_{t\in\mathbb{C}^r}$ is a non-trivial holomorphic deformation of germs at p of real-analytic CR mappings from M_1 to $\mathbb{T}^{m+1}$ (See [Mi] for the definition). Moreover, by (5.7) the real rank of (the Jacobian of) Ψ_0 (with respect to $(\mathrm{Re}z, \mathrm{Im}z)-$variables) at p is equal to that of F at p, which is at least 4 by the normalization (2.3). However, this contradicts with Lemma 2.3 of [Mi], which asserts that any holomorphic deformation of CR maps from M_1 to $\mathbb{T}^{m+1}$ must have real rank ≤ 2. Hence we have proved the claim in (5.8). For $(z,\tilde{w}) \in M_1\times V$, set

$$\rho^*(z,\overline{z},\tilde{w},\overline{\tilde{w}}) = \rho(\tilde{w},A(z,\tilde{w}),\overline{\tilde{w}},\overline{A(z,\tilde{w})}). \tag{5.9}$$

The algebraicity of A implies that $\rho^*(z,\xi,\tilde{w},\chi)$ is algebraic in its variables. We again have two cases.

Case 1: There exists a multi-index $\beta \in \mathbb{N}^r$ such that $\rho^*_{\tilde{w}^\beta}(z,\overline{z},\tilde{F}(z),\overline{\tilde{F}}(z)) \not\equiv 0$ on M_1. Let β_0 be the multi-index with the minimal $|\beta|$ such that the property holds. It follows from (5.7) that $|\beta_0| > 0$. Then as in the previous case 1, the conclusion follows from the algebraic version of the implicit function theorem.

Case 2: For any $\beta \in \mathbb{N}^r$, $\rho^*_{\tilde{w}^\beta}(z,\overline{z},\tilde{F}(z),\overline{\tilde{F}}(z)) \equiv 0$ on M_1. This implies $\rho^*(z,\overline{z},\tilde{w},\overline{\tilde{F}}(z)) \equiv 0$ for $(z,\tilde{w}) \in M_1\times V$. The equations (5.8) and (5.9) imply there exists some $\nu \in \mathbb{N}^\gamma$ with $|\nu| > 0$ such that $\rho^*_{\overline{\tilde{w}}^\nu}(z,\overline{z},\tilde{w},\overline{\tilde{F}}) \not\equiv 0$ on $M_1\times V$. The claim again is a consequence of the implicit function theorem.

We have thus established the claim. ∎

Observe (5.4) implies the assumption (5.5) holds in the claim with $r = n+1, l = 1$. Then apply the claim inductively $m-n$ times, we obtain there exists some open piece M_2 of M_0 containing p^* and a $\mathbb{C}^{m+1}-$valued algebraic function Δ defined in a neighborhood of $(p^*, \overline{p^*}, (L^\alpha \overline{h}(p^*))_{|\alpha| \leq m-n+1})$ such that

$$F(z) = \Delta(z, \overline{z}, (L^\alpha \overline{F}(z))_{|\alpha| \leq m-n+1}), z \in M_2.$$

We complexify the above equation by replacing $\overline{z}$ by the conjugate $\overline{\xi}$ of an independent variable ξ and fix $\xi = \xi_0 \approx p^* \in M_2$. This implies $F(z)$ is algebraic when restricted on Q_{ξ_0}. As ξ_0 is arbitrary, by Proposition 3.2, we conclude F is algebraic and thus establish Lemma 5.3.

It remains to show F is real-analytic on some open subset of M. This can be shown using the idea in [Mi]. Note a version of Claim 1 can be established in a similar manner where the algebraicity of Θ and η is replaced by analyticity. To show this version of Claim 1, one only needs to use finite smoothness of F rather than the analyticity. The readers are referred to Lemma 2.3 in [Mi] for more details. Once this version of Claim 1 is established, we apply it to the equation (5.4) inductively $m-n$ times to obtain, similarly as above, that

$$F(z) = \widetilde{\Delta}(z, \overline{z}, (L^\alpha \overline{F}(z))_{|\alpha| \leq m-n+1}), z \in M_2,$$

where M_2 is some open piece of M and $\widetilde{\Delta}(z, \xi, W)$ is analytic in z, ξ and W. Then by standard reflection principle, F extends holomorphically in a neighborhood of M_2. □

Proof of Proposition 1.5, part (2): Let H be the biholomorphism from τ_{m+1}^+ to D_{m+1}^{IV} mentioned above. Note H extends holomorphically across a generic boundary point of $\partial\tau_{m+1}$. Write H^{-1} be the inverse map of H. By shrinking M and composing F with some automorphism of D_{m+1}^{IV} if necessary, we can assume H^{-1} extends to a biholomorphism in a neighborhood of the image $F(M)$. Then the induced map $G := H^{-1} \circ F$ is C^{m-n+1} on M and maps M to the smooth part $\mathbb{T}^{m+1}$ of $\partial\tau_{m+1}$. Moreover, G is transversal on M as H^{-1} is a local biholomorphism. By Lemma 5.3, G extends to an algebraic map. We finally conclude F also extends to an algebraic map $H \circ G$. ■

Proof of Theorem 3: Fix $p_0 \in \Omega$. By composing an automorphism of D_{m+1}^{IV} if necessary, we assume $F(p_0) = 0$. Write M as the open piece of $\partial\Omega$ where F has C^2 smooth extension. As Ω is a bounded domain with real analytic boundary, we conclude that $\partial\Omega$ is strictly pseudoconvex at generic points (cf. [DA]). We claim that F sends an open dense subset M^* of M to the smooth part of ∂D_{m+1}^{IV}. Suppose not, then F must send a strongly pseudoconvex piece M_0 of M to the singular part E of ∂D_{m+1}^{IV}, which is given by $E = \{Z \in \mathbb{C}^{m+1} : 1 - ||Z||^2 + \frac{1}{4}|ZZ^t|^2 = 0, ||Z||^2 = 2\}$. Or equivalently, $E = \{Z \in \mathbb{C}^{m+1} : ||Z||^2 = 2, |ZZ^t| = 2\}$. Thus $h := FF^t = \sum_{j=1}^{m+1} F_j^2$ has constant modulus on M_0. This yields h must be constant on M_0. (cf. Theorem 5.1 in [BX1]). This further implies h is constant in Ω. This is a contradiction as $h(p_0) = 0$ and $|h| = 2$ on M_0. Hence F maps a dense open subset M^* of M to the smooth part of ∂D_{m+1}^{IV}. As D_{m+1}^{IV} is convex, by Proposition 5.1 we conclude that F is CR transversal along M^*. By part (2) of Proposition 1.5, F is algebraic. ■

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Department of Mathematics, University of California San Diego, La Jolla, CA 92093, USA.
E-mail address: m3xiao@ucsd.edu