Holomorphic isometric maps from the complex unit ball to reducible bounded symmetric domains

Ming Xiao *

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Abstract

The first part of the paper studies the boundary behavior of holomorphic isometric mappings $F = (F_1, \cdots, F_m)$ from the complex unit ball $\mathbb{B}^n, n \geq 2$, to a bounded symmetric domain $\Omega = \Omega_1 \times \cdots \times \Omega_m$ up to constant conformal factors, where Ω_i's are irreducible factors of Ω. We prove every non-constant component F_i must map generic boundary points of $\mathbb{B}^n$ to the boundary of Ω_i. In the second part of the paper, we establish a rigidity result for local holomorphic isometric maps from the unit ball to a product of unit balls and Lie balls.

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1 Introduction

The study of the rigidity and extension problem for holomorphic isometric maps goes back to the classical work of Calabi [Ca]. In 2003, motivated by problems in algebraic number theory, Clozel-Ullmo [CU] considered a local holomorphic isometric map from the Poincaré disk Δ into the polydisk Δ^p (each factor Δ is equipped with the Poincaré metric), and they proved that such a map must extend to a totally geodesic map providing the image is invariant under certain automorphisms of the target Δ^p. On the other hand, Mok [M3] shows the invariance assumption on the image of the map cannot be removed in this assertion. More

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precisely, Mok [M3] constructed a non-totally geodesic holomorphic isometric map, called the p-th root embedding, from the Poincaré disk Δ into the polydisk Δ^p. Furthermore, Mok initiated the systematic study of the local isometric mapping problems between bounded symmetric domains. See [M2], [M3], [M4] and references therein. In the following context, we write ds_D^2 for the Bergman metric of a bounded symmetric domain D. Let D_1, D_2 be two bounded symmetric domains and $V \subseteq D_1$ be an open connected set. Let $F : V \to D_2$ be a holomorphic isometric map in the sense that $F^*(ds^2_{D_2}) = \lambda ds^2_{D_1}$ for some positive constant λ. Mok [M3] proves that F extends to a holomorphic proper and isometric immersion from D_1 to D_2. Mok [M3] also proves F must be totally geodesic if D_1 is irreducible and has rank at least two.

Much less is known when the rank of D_1 equals one, i.e., D_1 is the complex unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ for some $n \geq 1$. It is natural to first study the local holomorphic isometric mappings from the unit ball $\mathbb{B}^n$ into the product of unit balls. The problem of holomorphic isometric maps from the Poincaré disk into polydisks were intensively studied by many authors. The readers are referred to Mok [M3], Ng [Ng1], Chan [Ch1, Ch2], Chan-Yuan [CY], Chan-Xiao-Yuan [CXY] and references therein. The current article will concentrate on the case $n \geq 2$. The problem in this case was studied by Mok [M2], Ng [Ng2] and Yuan-Zhang [YZ]. Let $F = (F_1, \cdots, F_m)$ be a holomorphic map from an open connected set $V \subseteq \mathbb{B}^n, n \geq 2$, to the product of unit balls $(\mathbb{B}^{N_1}, \lambda_1 ds^2_{\mathbb{B}^{N_1}}) \times \cdots \times (\mathbb{B}^{N_m}, \lambda_m ds^2_{\mathbb{B}^{N_m}})$ satisfying the metric-preserving property that $ds^2_{\mathbb{B}^n} = \sum_{i=1}^m \lambda_i F_i^*(ds^2_{\mathbb{B}^{N_i}})$ on V. Here λ_i's are positive constants. It follows from Yuan-Zhang [YZ] that the non-constant components F_i of F must extend to a totally geodesic map from $\mathbb{B}^n$ to $\mathbb{B}^{N_i}$ (The paper [YZ] indeed deals with a very general case where the λ_i's are allowed to be positive smooth Nash algebraic functions).

When D_1 is the unit ball and D_2 has an irreducible factor of rank at least two, the total geodesy rigidity of F fails dramatically. Mok [M4] constructed a non-totally geodesic holomorphic isometric map from $\mathbb{B}^n$ to a higher rank irreducible bounded symmetric domain D_2 of sufficiently large dimension (see also [XY1] for explicit examples of this kind). After the work of Mok [M4], many authors took the study of holomorphic isometric or proper maps from the unit ball to bounded symmetric domains of higher rank. See the work of Chan-Mok [CM], Xiao-Yuan [XY1, XY2], Upmeier-Wang-Zhang [UWZ], Chan [Ch3], etc. For more related study on metric-preserving or measure-preserving mappings, the readers are referred to [MN], [HY], [Y1, Y2], [FHX] and references therein.

Although the strong rigidity of total geodesy fails when D_2 has an irreducible factor of higher rank, it is believed by researchers that some weaker rigidity can still be expected. An explicit conjecture of this weaker rigidity was formulated by Yuan [Y2] which asserts that if $F = (F_1, \cdots, F_m)$ is a holomorphic isometric map from $(\mathbb{B}^n, ds^2_{\mathbb{B}^n}), n \geq 2$, to a reducible bounded symmetric domain $(\Omega_1, \lambda_1 ds^2_{\Omega_1}) \times \cdots \times (\Omega_m, \lambda_m ds^2_{\Omega_m})$ for some positive constants λ_i's, then the non-constant components F_i of F must be isometric (see more details in Problem

5.2, [Y2]). Although some partial results were proved when the dimension of Ω_i is not too much larger than n (cf. [XY2]), very little was known in the general case. The first step toward understanding the map F is to study its boundary behavior. We carry this out in Theorem 1.1. Then in Theorem 1.3 we confirm the weaker rigidity conjecture when the Ω_i's are either the unit ball or the type IV classical domain. To the best of our knowledge, this is the first theorem on this weaker rigidity conjecture that allows the dimension of Ω_i to be arbitrarily larger than n.

To introduce our theorems, we first recall some definitions and notations. We start with the notion of generic norms. Let D be an irreducible bounded symmetric domain and denote by $K_D(Z, \overline{Z})$ its Bergman kernel. Then there is a Hermitian polynomial $Q_D(Z, \overline{Z})$ such that $K_D(Z, \overline{Z}) = \frac{1}{Q_D(Z,\overline{Z})}$. Moreover, $Q_D(Z, \overline{Z}) = A_D \rho(Z, \overline{Z})^n$, where A_D is a positive constant, n is a positive integer both depending on D. Moreover, $\rho(Z, W)$ is an irreducible holomorphic polynomial satisfying $\rho(Z, \overline{Z}) > 0$ in D and $\rho(Z, \overline{Z}) = 0$ on the boundary ∂D, as well as $\rho(0, 0) = 1$. In addition, the expansion of $\rho(Z, \overline{Z}) - 1$ at $Z = 0$ has no pure terms. See [M1], [FK], [Lo] for more details on K_D and ρ. The function ρ is called the generic norm of D.

Throughout the paper, for an irreducible bounded symmetric domain D in some complex Euclidean space, we write g_D for the canonical complete Kähler-Einstein metric on D normalized so that the minimal disks are of constant Gaussian curvature -2. We denote by ω_D the corresponding Kähler form. Let Ω be a bounded symmetric domain and write $\Omega = \Omega_1 \times \cdots \times \Omega_m$. Here $\Omega_i, 1 \leq i \leq m$, is an irreducible bounded symmetric domain in some $\mathbb{C}^{N_i}$. Denote by $(\Omega, \oplus_{i=1}^m \lambda_i g_{\Omega_i}) = (\Omega_1, \lambda_1 g_{\Omega_1}) \times \cdots \times (\Omega_m, \lambda_m g_{\Omega_m})$ the bounded symmetric domain Ω equipped with the metric $\oplus_{i=1}^m \lambda_i g_{\Omega_i}$, where λ_i's are positive constants.

Let V be an open connected subset of the $n-$dimensional complex unit ball $\mathbb{B}^n$. Let $F = (F_1, \cdots, F_m)$ be a holomorphic map from V to $\Omega = \Omega_1 \times \cdots \times \Omega_m$, where each F_j maps V to Ω_j. We say F is a holomorphic isometric map from V to $(\Omega, \oplus_{i=1}^m \lambda_i g_{\Omega_i})$ if F preserves the metric in the following sense:

$$g_{\mathbb{B}^n} = \sum_{i=1}^m \lambda_i F_i^*(g_{\Omega_i}) \text{ in } V. \tag{1.1}$$

By Mok [M3] and Chan-Xiao-Yuan [CXY] (see Theorem 2.1.2 in [M3] and Theorem 4.25 in [CXY]), such a holomorphic isometric map F on V must be algebraic and extends to a holomorphic proper and isometric immersion from $\mathbb{B}^n$ to Ω. Thus it suffices to study global holomorphic isometric maps from $\mathbb{B}^n$ to Ω, and we can just assume $V = \mathbb{B}^n$. Recall F is called algebraic if each component $f_{i,l}$ of every F_i satisfies $P_{il}(z, f_{i,l}(z)) \equiv 0$ in V for some (nontrivial) irreducible polynomial $P_{il}(z, X)$ in $(z, X) \in \mathbb{C}^n \times \mathbb{C}$.

In this paper, we say F (which is defined on $\mathbb{B}^n$) extends holomorphically to (or, can be holomorphically continued to) some $p \in \partial\mathbb{B}^n$, if there exist a domain U containing $\mathbb{B}^n \cup \{p\}$, and a holomorphic map $\hat{F}$ on U satisfying $\hat{F} = F$ on $\mathbb{B}^n$. By the algebraicity of F, there is a complex hypervariety E in $\mathbb{C}^n$ such that F can be holomorphically continued along every path $\gamma \subset \mathbb{C}^n \backslash E$ with its initial point in $\mathbb{B}^n$. In particular, F extends holomorphically to every point $p \in \partial\mathbb{B}^n \setminus E$. Our first theorem describes the boundary behavior of each component F_i. Here for a nonzero real analytic function $h(z, \overline{z})$ defined on an open set $W \subseteq \mathbb{C}^n$, we say $h(z, \overline{z})$ has vanishing order $k \geq 0$ at $q \in W$ if the lowest nonzero term(s) in the Taylor expansion of h at q is of the form $\sum_{|\alpha|+|\beta|=k} c_{\alpha\beta}(z-q)^\alpha \overline{(z-q)}^\beta$.

Theorem 1.1. *Let $\Omega_i \subset \mathbb{C}^{N_i}, 1 \leq i \leq m$, be an irreducible bounded symmetric domain. Let $F = (F_1, \cdots, F_m)$ be a holomorphic isometric map from $(\mathbb{B}^n, g_{\mathbb{B}^n})$ to $(\Omega_1, \lambda_1 g_{\Omega_1}) \times \cdots \times (\Omega_m, \lambda_m g_{\Omega_m})$ satisfying $g_{\mathbb{B}^n} = \sum_{i=1}^m \lambda_i F_i^*(g_{\Omega_i})$ in $\mathbb{B}^n$. Here λ_i's are positive constants. Write $\mathcal{S}$ for the set of points $p \in \partial\mathbb{B}^n$ to which F extends holomorphically. (Then $\mathcal{S}$ is open in $\partial\mathbb{B}^n$ and by the above discussion, there is complex hypervariety E in $\mathbb{C}^n$ satisfying $\partial\mathbb{B}^n \setminus E \subseteq \mathcal{S}$.) Assume $n \geq 2$ and every F_i is non-constant. Then the following two conclusions hold:*

- *(a). For every $p \in \mathcal{S}$, the holomorphic continuation of each F_i to p, which is still denoted by F_i, must map p to $\partial\Omega_i$.*
- *(b). Denote by ρ_i the generic norm of Ω_i. For each $1 \leq i \leq m$, there exists some integer $k_i \geq 1$, such that the vanishing order of $\rho_i(F_i, \overline{F_i})$ at every $p \in \mathcal{S}$ equals k_i. Moreover, it holds that $\sum_{i=1}^m k_i\lambda_i = 1$.*

Remark 1.2. *Note Theorem 1.1 is optimal in the sense that the assumption of $n \geq 2$ cannot be removed. Indeed, with $p \geq 2$, Mok's p-th root map (see page 1648, [M3]) gives an example of holomorphic isometric embedding $F = (F_1, \cdots, F_p)$ from the Poincaré disk Δ into the polydisk Δ^p, where for every $1 \leq i \leq p$, the holomorphic continuation of F_i maps some open piece of $\partial\Delta$ to Δ. Hence the assertion in Theorem 1.1 fails when $n = 1$.*

We will give in Section 2 a refined version of Theorem 1.1 for the case when Ω is a product of Cartan's classical domains (see Theorem 2.3). Theorem 1.1 makes it possible to apply machinery from CR geometry to study holomorphic isometric maps from the unit ball to bounded symmetric domains. In particular, we will apply Theorem 1.1, as well as recently developed techniques in CR geometry, to study isometric maps from the unit ball to the product of unit balls and Lie balls. Recall the type IV classical domain D_N^{IV} in $\mathbb{C}^N (N \geq 2)$, also called the Lie ball, is defined by

$$D_N^{IV} = \{Z = (z_1, \cdots, z_N) \in \mathbb{C}^N : Z\overline{Z}^t < 2 \text{ and } 1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2 > 0\}.$$

The Kähler form $\omega_{D_N^{IV}}$ associated to the Kähler-Einstein metric $g_{D_N^{IV}}$ is given by

$$\omega_{D_N^{IV}} = -\sqrt{-1}\partial\overline{\partial}\log(1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2). \tag{1.2}$$

In the second part of the paper, we will establish the following rigidity result.

Theorem 1.3. *Let $\Omega_i, 1 \leq i \leq m$, be either the complex unit ball $\mathbb{B}^{N_i}$ for some $N_i \geq 1$ or the Lie ball $D_{N_i}^{IV}$ for some $N_i \geq 2$. Let $F = (F_1, \cdots, F_m)$ be a holomorphic isometric map from an open connected set $V \subseteq \mathbb{B}^n$ to $(\Omega_1, \lambda_1 g_{\Omega_1}) \times \cdots \times (\Omega_m, \lambda_m g_{\Omega_m})$ satisfying (1.1), where $\lambda_i's$ are positive constants. Assume $n \geq 4$ and each F_i is non-constant. Then every $F_i, 1 \leq i \leq m$, extends to a holomorphic isometric embedding from $\mathbb{B}^n$ to Ω_i with $F_i^*(g_{\Omega_i}) = g_{\mathbb{B}^n}$. Furthermore, $\sum_{i=1}^m \lambda_i = 1$.*

One key idea to prove Theorem 1.3 is to realize $D_{N_i}^{IV}$ as an isometric submanifold in the indefinite hyperbolic space $\mathbb{B}_1^{N_i+1}$ (see Section 3 for the definition of the latter). With the help of Theorem 1.1, we will show each F_i naturally induces a local holomorphic map that sends an open piece of $\partial\mathbb{B}^n$ to $\partial\mathbb{B}_1^{n_i+1}$, and F induces an isometric map from $\mathbb{B}^n$ to a product of indefinite hyperbolic spaces. To study the induced maps, we apply recently developed ideas and methods in CR geometry. Especially the work in [HLTX1] will play a fundamental role in the proof. We also borrow ideas from the work of Yuan-Zhang [YZ]. We should mention that, in Theorem 1.3 when Ω is just a single copy of a Lie ball, classification and characterization results of the map were established in [CM], [XY2] and [UWZ]. In particular, by combining the results in [CM] and [XY2], we see any holomorphic isometric map $H : \mathbb{B}^n \to D_N^{IV}, N > n \geq 2$, can be decomposed into the form $H = \varphi \circ f \circ \tau \circ i \circ \sigma$. Here σ, τ, φ are automorphisms of $\mathbb{B}^n, \mathbb{B}^{N-1}, D_N^{IV}$, respectively. The map i is the standard linear embedding from $\mathbb{B}^n$ to $\mathbb{B}^{N-1}$. And f is either of the two maps from $\mathbb{B}^{N-1}$ to D_N^{IV} as defined in Theorem 1.2 of [XY2], one of which is rational and the other irrational.

The paper is organized as follows. Section 2 is devoted to establishing Theorem 1.1. In Section 3, as a preparation for the proof of Theorem 1.3, we study local holomorphic isometric maps from the unit ball to the product of indefinite hyperbolic spaces under some boundary conditions. In Section 4, we use Theorem 1.1 and results in Section 3 to prove Theorem 1.3.

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2 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. Given $p \in \mathcal{S}$, we fix a small ball O in $\mathbb{C}^n$ centered at p such that F extends holomorphically to O (note $\partial\mathbb{B}^n \cap O$ is connected). We still denote the extension by $F = (F_1, \cdots, F_m)$.

Write Z_i for the coordinates of $\mathbb{C}^{N_i}$. By the definition of the metric g_{Ω_i} on the irreducible bounded symmetric domain $\Omega_i \subset \mathbb{C}^{N_i}$, the corresponding Kähler form is given by:

$$\omega_{\Omega_i} = -\sqrt{-1}\partial\overline{\partial}\log \rho_i(Z_i, \overline{Z}_i).$$

Here ρ_i denotes the generic norm of Ω_i. By composing F with automorphisms of Ω if necessary, we can assume $F(0) = 0$. Write $|\cdot|$ for the Euclidean norm. By using the metric preserving assumption (1.1), properties of ρ_i, and a standard reduction (see for example [M3], [HY]), we have

$$1 - |z|^2 = \prod_{i=1}^{m} \big(\rho_i\big(F_i, \overline{F_i}\big)\big)^{\lambda_i} \text{ on } \mathbb{B}^n. \tag{2.1}$$

Letting $z \in \mathbb{B}^n \to \partial\mathbb{B}^n \cap O$ in (2.1) (or by the properness of F), we see there exists some $1 \leq i \leq m$, such that $\rho_i\big(F_i(z), \overline{F_i(z)}\big) \equiv 0$ on $\partial\mathbb{B}^n \cap O$, or equivalently, $F_i(\partial\mathbb{B}^n \cap O) \subseteq \partial\Omega_i$ (here we have used the fact that F_i is real analytic on $\partial\mathbb{B}^n \cap O$ and that $\partial\mathbb{B}^n \cap O$ is connected). The following lemma shows it is indeed the case for all F_i.

Proposition 2.1. *Under the assumptions of Theorem 1.1 and the above notations, for every $1 \leq i \leq m$, F_i maps $\partial\mathbb{B}^n \cap O$ to $\partial\Omega_i$.*

Proof of Proposition 2.1: First by the preceding discussion, there is at least one i such that F_i maps $\partial\mathbb{B}^n \cap O$ to $\partial\Omega_i$. Then after re-ordering $F_i's$ and Ω_i's, we can find some $1 \leq i_0 \leq m$ such that the following two conditions hold:

(I). For every $1 \leq i \leq i_0$, we have $F_i(\partial\mathbb{B}^n \cap O) \subseteq \partial\Omega_i$.

(II). There is a smaller ball $\hat{O} \subset O$ in $\mathbb{C}^n$ centered at some $\hat{q} \in \partial\mathbb{B}^n$ near p such that, for every $i_0 + 1 \leq i \leq m, F_i$ maps every $q \in \hat{O}$ to Ω_i. Consequently, $\rho_i\big(F_i(z), \overline{F_i(z)}\big) > 0$ in $\hat{O}$.

To establish Proposition 2.1, it suffices to show $i_0 = m$. Seeking a contradiction, we suppose $i_0 < m$. We first note for each $1 \leq i \leq i_0$, there exists an integer $k_i \geq 1$ and a real analytic function ψ_i in O such that $\psi_i \not\equiv 0$ on $\partial\mathbb{B}^n \cap O$, and

$$\rho_i\big(F_i(z), \overline{F_i(z)}\big) = (1 - |z|^2)^{k_i}\psi_i(z, \overline{z}) \text{ in } O, \quad 1 \leq i \leq i_0. \tag{2.2}$$

By further shrinking $\hat{O}$ to a smaller ball centered at some $\widetilde{q} \in \partial\mathbb{B}^n$ near $\hat{q}$ if necessary, we can assume $\psi_i(z, \overline{z}) \neq 0$ everywhere in $\hat{O}$ for every $1 \leq i \leq i_0$. Furthermore, since $1 - |z|^2 > 0$ and $\rho_i\big(F_i(z), \overline{F_i(z)}\big) > 0$ on $\mathbb{B}^n \cap \hat{O}$, it follows that for $1 \leq i \leq i_0$, $\psi_i(z, \overline{z})$ is everywhere positive in $\mathbb{B}^n \cap \hat{O}$, and thus also positive in $\hat{O}$. Next by (2.1) and (2.2), we have

$$1 - |z|^2 = \prod_{i=1}^{i_0}(1 - |z|^2)^{k_i\lambda_i}\big(\psi_i(z, \overline{z})\big)^{\lambda_i} \prod_{i=i_0+1}^{m} \big(\rho_i\big(F_i(z), \overline{F_i(z)}\big)\big)^{\lambda_i} \text{ in } \mathbb{B}^n \cap O. \tag{2.3}$$

Note in $\hat{O}, \psi_i(z,\overline{z}) > 0$ for all $1 \leq i \leq i_0$; and $\rho_i\big(F_i(z), \overline{F_i(z)}\big) > 0$ for all $i_0+1 \leq i \leq m$. Letting $z \in \mathbb{B}^n \cap \hat{O} \to \partial\mathbb{B}^n \cap \hat{O}$ in (2.3) and comparing the vanishing order of both sides, we see that

$$\sum_{i=1}^{i_0} k_i \lambda_i = 1. \tag{2.4}$$

Then (2.3) is reduced to

$$1 = \prod_{i=1}^{i_0} \big(\psi_i(z,\overline{z})\big)^{\lambda_i} \prod_{k=i_0+1}^{m} \big(\rho_k\big(F_k, \overline{F_k}\big)\big)^{\lambda_k} \text{ in } \mathbb{B}^n \cap \hat{O}. \tag{2.5}$$

Note for any $\lambda > 0$, the function $h(y) = y^\lambda$ is real analytic on $(0,\infty)$. By the positivity and real analycity of $\psi_i, 1 \leq i \leq i_0$, and $\rho_k\big(F_k, \overline{F_k}\big), i_0+1 \leq k \leq m$, on the ball $\hat{O}$, we see the right hand side of (2.5) is real analytic in $\hat{O}$. Consequently, (2.5) indeed holds in $\hat{O}$:

$$1 = \prod_{i=1}^{i_0} \big(\psi_i(z,\overline{z})\big)^{\lambda_i} \prod_{k=i_0+1}^{m} \big(\rho_k\big(F_k, \overline{F_k}\big)\big)^{\lambda_k} \text{ in } \hat{O}. \tag{2.6}$$

Next we define $r_i(Z_i, \overline{Z}_i) := (-1)^{k_i} \rho_i(Z_i, \overline{Z}_i)$ for $1 \leq i \leq i_0$. It then follows from (2.2) that

$$r_i\big(F_i(z), \overline{F_i(z)}\big) = (|z|^2 - 1)^{k_i} \psi_i(z,\overline{z}) \text{ in } O, \text{ in particular in } \hat{O}, \quad 1 \leq i \leq i_0. \tag{2.7}$$

The above implies $r_i\big(F_i(z), \overline{F_i(z)}\big) > 0$ in $\hat{O} \setminus \overline{\mathbb{B}^n}$ for $1 \leq i \leq i_0$. Moreover, by (2.6) and (2.7),

$$1 = \prod_{i=1}^{i_0} \left(\frac{r_i\big(F_i, \overline{F_i}\big)}{(|z|^2-1)^{k_i}} \right)^{\lambda_i} \prod_{k=i_0+1}^{m} \big(\rho_k\big(F_k, \overline{F_k}\big)\big)^{\lambda_k} \text{ in } \hat{O} \setminus \overline{\mathbb{B}^n}. \tag{2.8}$$

Equivalently, we have

$$|z|^2 - 1 = \prod_{i=1}^{i_0} \big(r_i\big(F_i, \overline{F_i}\big)\big)^{\lambda_i} \prod_{k=i_0+1}^{m} \big(\rho_k\big(F_k, \overline{F_k}\big)\big)^{\lambda_k} \text{ in } \hat{O} \setminus \overline{\mathbb{B}^n}. \tag{2.9}$$

Recall F, which is in particular holomorphic in $\hat{O}$, extends holomorphically along any path γ in $\mathbb{C}^n \setminus E$ for some complex hypervariety E in $\mathbb{C}^n$. (For convenience, by further shrinking $\hat{O}$ if necessary, we can assume $\hat{O} \cap E = \emptyset$.) We will still denote the holomorphic continuation of F_i along γ by $F_i, 1 \leq i \leq m$. We have the following lemma regarding the continuation. To simplify the notations, we write Γ for the set of all paths $\gamma : [0,1] \to \mathbb{C}^n \setminus (\overline{\mathbb{B}^n} \cup E)$ with $\gamma(0) \in \hat{O} \setminus \overline{\mathbb{B}^n}$. When we say $\gamma \in \Gamma$, we always assume γ is parameterized over the interval $[0,1]$.

Lemma 2.2. *(1). For every path $\gamma \in \Gamma$, and every $1 \leq i \leq m$, we have $\rho_i(F_i, \overline{F_i})$ is nonzero along γ :*

$$\rho_i(F_i(\gamma(t)), \overline{F_i(\gamma(t))}) \neq 0, \text{ for } 0 \leq t \leq 1.$$

(2). Fix any $i_0 + 1 \leq k \leq m$. Then for every path $\gamma \in \Gamma$, we have $F_k(\gamma(t)) \in \Omega_k$ holds for $t \in [0, 1]$. In particular, there is a positive constant M_k(only depending on Ω_k) such that $|F_k(z)| \leq M_k$ along every $\gamma \in \Gamma$.

Proof of Lemma 2.2: We prove part (1) of the lemma by contradiction. Suppose not. Then there is a path $\gamma \in \Gamma$ such that when F is continued holomorphically along γ, we have

$$\prod_{i=1}^{m} \rho_i(F_i, \overline{F_i}) = 0 \text{ at the point } z = \gamma(1).$$

This yields that

$$\prod_{i=1}^{i_0} r_i\big(F_i, \overline{F_i}\big) \prod_{k=i_0+1}^{m} \rho_k\big(F_k, \overline{F_k}\big) = 0 \text{ at the point } z = \gamma(1).$$

Recall that we have $r_i\big(F_i(z), \overline{F_i(z)}\big) > 0$ in $\hat{O} \setminus \mathbb{B}^n$ for $1 \leq i \leq i_0$; and $\rho_k\big(F_k, \overline{F_k}\big) > 0$ in $\hat{O}$ for $i_0 + 1 \leq k \leq m$. Now set

$$t_0 = \sup\left\{ t > 0 : \prod_{i=1}^{i_0} r_j\big(F_i, \overline{F_i}\big) \prod_{k=i_0+1}^{m} \rho_k\big(F_k, \overline{F_k}\big) > 0 \text{ along } \gamma([0, t)) \right\}.$$

It is clear that $0 < t_0 \leq 1$. And $\prod_{i=1}^{i_0} r_i\big(F_i, \overline{F_i}\big) \prod_{k=i_0+1}^{m} \rho_k\big(F_k, \overline{F_k}\big) = 0$ at $z = \gamma(t_0)$. Moreover,

$$r_i\big(F_i, \overline{F_i}\big) > 0 \text{ along } \gamma([0, t_0)) \text{ for } 1 \leq i \leq i_0;$$

$$\rho_k\big(F_k, \overline{F_k}\big) > 0 \text{ along } \gamma([0, t_0)) \text{ for } i_0 + 1 \leq k \leq m.$$

Note $\big(r_i\big(F_i, \overline{F_i}\big)\big)^{\lambda_i}$ is real analytic wherever $r_i\big(F_i, \overline{F_i}\big)$ is a positive real analytic function. Likewise for $\big(\rho_k\big(F_k, \overline{F_k}\big)\big)^{\lambda_k}$. It then follows from the analyticity that (2.9) holds along $\gamma([0, t_0))$. That is, for $0 \leq t < t_0$,

$$|\gamma(t)|^2 - 1 = \prod_{i=1}^{i_0} \Big(r_i\big(F_i(\gamma(t)), \overline{F_i(\gamma(t))}\big) \Big)^{\lambda_i} \prod_{k=i_0+1}^{m} \big(\rho_k\big(F_k(\gamma(t)), \overline{F_k(\gamma(t))}\big)\big)^{\lambda_k}. \tag{2.10}$$

Letting $t \to t_0-$ on both sides of (2.10), we have the limit of the left hand side is positive as $\gamma \subset \mathbb{C}^n \setminus (\overline{\mathbb{B}^n} \cup E)$, while the limit of the right hand side equals 0. This is a contradiction. Part (1) of Lemma 2.2 is thus proved.

We also prove part (2) of Lemma 2.2 by contradiction. Fix $i_0 + 1 \leq k \leq m$. Suppose the first assertion of part (2) does not hold. Then there exists a path $\gamma \in \Gamma$ such that the holomorphic continuation of F_k along γ, which is still denoted by F_k, satisfies

$$F_k(\gamma(1)) \in \mathbb{C}^{N_k} \setminus \Omega_k.$$

We recall $F_k(z) \in \Omega_j$ for all $z \in \hat{O}$, in particular $F_k(\gamma(0)) \in \Omega_k$. Since $\partial\Omega_k$ separates the two connected open subsets Ω_k and $\mathbb{C}^{N_k} \setminus \overline{\Omega_k}$ in $\mathbb{C}^{N_k}$, there exists some $0 < t^* \leq 1$ such that $F_k(\gamma(t^*)) \in \partial\Omega_k$. But this implies $\rho_k\big(F_k, \overline{F_k}\big) = 0$ at the point $\gamma(t^*)$, a plain contradiction to part (1) of Lemma 2.2. This proves the first assertion in part (2). The second assertion in part (2) then immediately follows from the boundedness of Ω_k. Hence part (2) of Lemma 2.2 is also established. ■

We continue to prove Proposition 2.1. For $i_0 + 1 \leq k \leq m$, write $F_k = (f_{k,1}, \cdots, f_{k,N_k})$, which is in particular a holomorphic map from $\hat{O}$ to $\mathbb{C}^{N_k}$. For any fixed $i_0 + 1 \leq k \leq m, 1 \leq l \leq N_k$ and $q \in \mathbb{C}^n \setminus (\overline{\mathbb{B}^n} \cup E)$, write $\{(f_{k,l})_{j,q}\}_{j=1}^{\nu_{kl}}$ for all possible (distinct) germs of holomorphic functions at q that can be obtained by applying holomorphic continuation to $f_{k,l}$ along paths $\gamma \in \Gamma$. Let $\tau \geq 1$ and $\sigma_{k,l,\tau}$ be the fundamental symmetric function of $\{(f_{k,l})_{j,q}\}_{j=1}^{\nu_{kl}}$ of degree τ. Then $\sigma_{k,l,\tau}$ is a well-defined holomorphic function in $\mathbb{C}^n \setminus (\overline{\mathbb{B}^n} \cup E)$. Moreover, for each $i_0+1 \leq k \leq m, 1 \leq l \leq N_k$ and $\tau \geq 1$, $\sigma_{k,l,\tau}$ is bounded over $\mathbb{C}^n \setminus (\overline{\mathbb{B}^n} \cup E)$ by part (2) of Lemma 2.2. Thus by Riemann's removable singularity theorem, $\sigma_{k,l,\tau}$ extends to a bounded holomorphic function in $\mathbb{C}^n \setminus \overline{\mathbb{B}^n}$. Then by Hartogs's extension theorem (recall $n \geq 2$), $\sigma_{k,l,\tau}$ extends to a bounded holomorphic function in $\mathbb{C}^n$, which must be constant by Liouville's theorem. Finally, since every $\sigma_{k,l,\tau}$ is constant, we have $f_{k,l}$ must be constant function and therefore F_k is a constant map for every $i_0 + 1 \leq k \leq m, 1 \leq l \leq N_k$. This contradicts the assumption of Theorem 1.1 if $i_0 < m$. Hence we must have $i_0 = m$ and this finishes the proof of Proposition 2.1. ■

We should remark that the above idea of applying Hartogs's extension theorem and Liouville's theorem to study the extension of isometric maps shares the same spirit as that of [Ng2] and [YZ]. We are now at the position to prove Theorem 1.1.

Proof of part (a) in Theorem 1.1: It's clear that part (a) of Theorem 1.1 follows from Proposition 2.1 as O can be a small ball centered at an arbitrary point $p \in \mathcal{S}$. ■

Proof of part (b) in Theorem 1.1: To prove part (b) of Theorem 1.1, we first note (2.3) is now reduced to the following (recall we have proved $i_0 = m$):

$$1-|z|^2=\prod_{i=1}^{m}(1-|z|^2)^{k_i\lambda_i}\big(\psi_i(z,\overline{z})\big)^{\lambda_i} \text{ on } \mathbb{B}^n\cap O; \tag{2.11}$$

where we recall $\psi_i's$ are real analytic functions on O. Similarly, (2.4) is reduced to $\sum_{i=1}^m k_i\lambda_i = 1$. Combining this with (2.11), we have

$$1=\prod_{i=1}^{m}\big(\psi_i(z,\overline{z})\big)^{\lambda_i} \text{ on } \mathbb{B}^n\cap O. \tag{2.12}$$

Here we recall by (2.2), $\psi_i(z,\overline{z})>0$ on $\mathbb{B}^n\cap O$. Fix any $q\in\partial\mathbb{B}^n\cap O$. Let $z\in\mathbb{B}^n\cap O\to q$, we get $\psi_i(q,\overline{q})\neq 0$. By using (2.2) and checking the Taylor expansion of the right hand side of (2.2) at q, we see the vanishing order of $\rho_i(F_i,\overline{F_i})$ at q equals k_i for every $q\in\partial\mathbb{B}^n\cap O$. Thus the vanishing order of $\rho_i(F_i,\overline{F_i})$ is locally constant on $\mathcal{S}$. Note $\partial\mathbb{B}^n\cap E$ is of real dimension at most $2n-3$. Consequently, $\partial\mathbb{B}^n\setminus E$ is connected. So is $\mathcal{S}$. Hence the vanishing order of $\rho_i(F_i,\overline{F_i})$ is constant on $\mathcal{S}$ (and equals k_i). This proves part (b) of Theorem 1.1. ■

At the end of this section, we provide a refined version of Theorem 1.1 in the case when Ω is a product of Cartan's classical domains. For that, we first recall some preliminary about the boundary structure of an irreducible bounded symmetric domain D. By Borel embedding (cf. [M1]), D can be canonically embedded into its dual Hermitian symmetric manifolds X of compact type. Under the embedding, every automorphism $g\in\mathrm{Aut}(D)$ extends to an automorphism of X and D becomes an open orbit under the action of $\mathrm{Aut}(D)$ on X. Moreover, denoting the rank of D by r, the topological boundary ∂D of D decomposes into exactly r orbits under the action of the identity component $\mathrm{Aut}_0(D)$ of $\mathrm{Aut}(D)$: $\partial D=\cup_{j=1}^r E_j$, where E_k lies in the closure of E_l if $k>l$. Moreover, E_k is the set of smooth points of the semi-analytic variety $\cup_{j=k}^r E_j$ (see the proof of Lemma 2.2.3 in [MN]). In particular, E_1 consists of the smooth points of ∂D.

Let $\Omega_i, 1\leq i\leq m$, be an (irreducible) Cartan's classical domain. By the above discussion, we can write the stratification of the boundary of Ω_i as

$$\partial\Omega_i=\cup_{l=1}^{r_i}E_{i,l} \text{ with } r_i=\mathrm{rank}(\Omega_i). \tag{2.13}$$

Here the $E_{i,l}'s$ are the orbits under the action of $\mathrm{Aut}_0(\Omega_i)$ as described in the above, satisfying that $E_{i,k}\subseteq\overline{E_{i,l}}$ for $k>l$.

Theorem 2.3. *Let $\Omega_i\subset\mathbb{C}^{N_i}, 1\leq i\leq m$, be an (irreducible) Cartan's classical domain with $\mathrm{rank}(\Omega_i)=r_i$ and with the boundary stratification (2.13). Let $F=(F_1,\cdots,F_m)$ be a holomorphic isometric map from $\mathbb{B}^n$ to $\Omega=(\Omega_1,\lambda_1 g_{\Omega_1})\times\cdots\times(\Omega_m,\lambda_m g_{\Omega_m})$ satisfying $g_{\mathbb{B}^n}=\sum_{j=1}^m\lambda_j F_j^*(g_{\Omega_j})$ in $\mathbb{B}^n$. Here $\lambda_i's$ are positive constants. Let $\mathcal{S}$ be as defined in*

Theorem 1.1. Assume $n \geq 2$ and every F_i is non-constant. Then for each $1 \leq i \leq m$, there exists some integer $1 \leq k_i \leq r_i$ such that for every $p \in \mathcal{S}$, the holomorphic continuation of F_i to p maps p to E_{i,k_i}. Moreover, $\sum_{i=1}^{m} k_i \lambda_i = 1$.

Proof of Theorem 2.3: Fix $1 \leq i \leq m$. By Theorem 1.1, the holomorphic extension of F_i to $p \in \mathcal{S}$ must map p to $\partial\Omega_i$. Furthermore, still denoting the extension by F_i, there exists some $k_i \geq 1$, such that the vanishing order of $\rho_i(F_i, \overline{F_i})$ at p equals k_i for some k_i independent of the choice of p. Moreover, $\sum_{i=1}^{m} k_i \lambda_i = 1$. By Theorem 1 in [X1], F_i maps the point $p \in \mathcal{S}$ to $E_{i,l}$ if and only if the vanishing order of $\rho_i(F_i, \overline{F_i})$ at p equals l. Consequently, $1 \leq k_i \leq r_i$. The other assertions in Theorem 2.3 follows as well. ■

3 Isometric maps into the product of indefinite hyperbolic spaces

In this section, as a preparation for the proof of Theorem 1.3, we study local holomorphic isometric mappings from the unit ball (the hyperbolic space) to the product of indefinite hyperbolic spaces. In §3.1, we first recall some basic definitions and preliminaries about indefinite hyperbolic spaces. The section §3.2 proves a couple of algebraic lemmas which will be used in the later proof. In §3.3, we prove a rigidity result for local holomorphic isometric mappings from the unit ball to the product of indefinite hyperbolic spaces. The rigidity result will be fundamentally used in the proof of Theorem 1.3.

3.1 Some preliminary of indefinite hyperbolic spaces

Let n, ℓ be integers such that $n \geq 2$ and $0 \leq \ell \leq n-1$. The generalized complex unit ball is defined as the following domain in $\mathbb{P}^n$:

$$\mathbb{B}^n_\ell = \{[z_0, ..., z_n] \in \mathbb{P}^n : |z_0|^2 + ... + |z_\ell|^2 > |z_{\ell+1}|^2 + ... + |z_n|^2\}.$$

In the special case of $\ell = 0$, $\mathbb{B}^n_0$ is reduced to the standard unit ball $\mathbb{B}^n$ (embedded in $\mathbb{P}^n$). The generalized ball $\mathbb{B}^n_\ell$ carries a canonical (pseudo-Kähler) metric $g_{\mathbb{B}^n_\ell}$ that is invariant under the action of its automorphism group $PSU(\ell+1, n+1)$, where the latter means the projectivization of $SU(\ell+1, n+1)$. The corresponding Kähler form $\omega_{\mathbb{B}^n_\ell}$ of $g_{\mathbb{B}^n_\ell}$ is given by

$$\omega_{\mathbb{B}^n_\ell} = -\sqrt{-1}\partial\bar{\partial}\log(\sum_{j=0}^{\ell}|z_j|^2 - \sum_{j=\ell+1}^{n}|z_j|^2). \quad (3.1)$$

Note the (pseudo-Kähler) metric $g_{\mathbb{B}^n_\ell}$ is indefinite if and only if $\ell \geq 1$. In this case, the generalized ball equipped with the metric $\omega_{\mathbb{B}^n_\ell}$ is called the indefinite hyperbolic space form. In the case $\ell = 0$, it is reduced to the standard hyperbolic space form (up to a normalization). To better perform the local CR differential analysis on the boundary, we also often work with a different realization of $\mathbb{B}^n_\ell$, which is known as (generalized) Siegel upper-half space. To introduce the latter, we first fix some notations that will be used throughout Section 3.

Given a fixed $\ell \geq 0$, we denote by $\delta_{j,\ell}$ the symbol which takes value -1 when $1 \leq j \leq \ell$ and 1 otherwise. If $l = 0$, $\delta_{j,0}$ is identically one for all $j \geq 1$. For fixed integers $\ell' \geq \ell \geq 1$ and $n \geq 1$, we denote by $\delta_{j,\ell,\ell',n}$ the symbol which takes value -1 when $1 \leq j \leq \ell$ or $n \leq j \leq n+\ell'-\ell-1$, and 1 otherwise. When $\ell' = \ell$, $\delta_{j,\ell,\ell',n}$ is the same as $\delta_{j,\ell}$. Let $m \geq 1$. For two m-tuples $x = (x_1, \cdots, x_m)$, $y = (y_1, \cdots, y_m)$ of complex numbers, we write $\langle x, y\rangle_\ell = \sum_{j=1}^m \delta_{j,\ell} x_j y_j$, and $|x|^2_\ell = \langle x, \bar{x}\rangle_\ell$. Also write $\langle x, y\rangle_{\ell,\ell',n} = \sum_{j=1}^m \delta_{j,\ell,\ell',n} x_j y_j$ and $|x|^2_{\ell,\ell',n} = \langle x, \bar{x}\rangle_{\ell,\ell',n}$. For $0 \leq \ell \leq n-1$, we define the generalized Siegel upper-half space

$$\mathbb{S}^n_\ell = \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \mathrm{Im}(w) > |z|^2_\ell\}.$$

When $l = 0$, it is reduced to the standard Siegel upper-half space. The topological boundary $\mathbb{H}^n_\ell$ of $\mathbb{S}^n_\ell$, called the generalized Heisenberg hypersurfaces, is defined by the equation $\mathrm{Im}(w) = |z|^2_\ell$. Now for $(z, w) = (z_1, \cdots, z_{n-1}, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$, let $\Psi_n(z, w) = [i+w, 2z, i-w] \in \mathbb{P}^n$. Then Ψ_n is the Cayley transformation which biholomorphically maps the generalized Siegel upper-half space $\mathbb{S}^n_\ell$ and its boundary $\mathbb{H}^n_\ell$ onto $\mathbb{B}^n_\ell \setminus \{[\xi_0, \cdots, \xi_n] : \xi_0 + \xi_n = 0\}$ and $\partial\mathbb{B}^n_\ell \setminus \{[\xi_0, \cdots, \xi_n] : \xi_0 + \xi_n = 0\}$, respectively.

We also define for $\ell \leq n-1$, $\ell \leq \ell' \leq N-1$ and $N \geq n + \ell' - \ell$,

$$\mathbb{S}^N_{\ell,\ell',n} = \{(z, w) \in \mathbb{C}^{N-1} \times \mathbb{C} : \mathrm{Im}(w) > |z|^2_{\ell,\ell',n}\}.$$

Note $\mathbb{S}^N_{\ell,\ell',n}$ is identical to $\mathbb{S}^N_{\ell'}$ if $\ell' = \ell$. When $\ell' > \ell$, $\mathbb{S}^N_{\ell,\ell',n}$ is holomorphically equivalent to $\mathbb{S}^N_{\ell'}$ by a permutation $\mathcal{P}$ of coordinates in $\mathbb{C}^N$. We will more often work with $\mathbb{S}^N_{\ell,\ell',n}$ instead of $\mathbb{S}^N_{\ell'}$, as it makes notations simpler. The topological boundary $\mathbb{H}^N_{\ell,\ell',n}$ of $\mathbb{S}^N_{\ell,\ell',n}$ is defined by the equation $\mathrm{Im}(w) = |z|^2_{\ell,\ell',n}$. Writing Ψ_N for the Cayley transformation which biholomorphically maps $\mathbb{S}^N_{\ell'}$ onto $\mathbb{B}^N_{\ell'} \setminus \{[\xi_0, \cdots, \xi_N] : \xi_0 + \xi_N = 0\}$, the map $\Psi^N_{\ell,\ell',n} := \Psi_N \circ \mathcal{P}$ biholomorphically maps $\mathbb{S}^N_{\ell,\ell',n}$ and its boundary $\mathbb{H}^N_{\ell,\ell',n}$ onto $\mathbb{B}^N_{\ell'} \setminus \{[\xi_0, \cdots, \xi_N] : \xi_0 + \xi_N = 0\}$ and $\partial\mathbb{B}^N_{\ell'} \setminus \{[\xi_0, \cdots, \xi_N] : \xi_0 + \xi_N = 0\}$, respectively. We will call $\Psi^N_{\ell,\ell',n}$ the generalized Cayley transformation.

Note the pull back of $g_{\mathbb{B}^n_\ell}$ by Ψ_n gives a canonical indefinite metric $g_{\mathbb{S}^n_\ell} = \Psi^*_n(g_{\mathbb{B}^n_\ell})$ on $\mathbb{S}^n_\ell$. Writing $(z_1, \cdots, z_{n-1}, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ for the coordinates of $\mathbb{C}^n$, the corresponding Kähler form $\omega_{\mathbb{S}^n_\ell}$ of $g_{\mathbb{S}^n_\ell}$ is given by $-\sqrt{-1}\partial\bar{\partial}\log\left(\mathrm{Im}(w) - |z|^2_l\right)$. We in particular recall here the explicit formula of $g_{\mathbb{S}^n_\ell}$ when $\ell = 0$. In this case, the metric $g_{\mathbb{S}^n_0}$ is indeed the normalized Bergman metric of the standard Siegel upper-half space $\mathbb{S}^n_0$.

$$
\begin{aligned}
g_{\mathbb{S}^n_0} = & \sum_{1\le j,k\le n-1} \frac{\delta_{jk}\big(\mathrm{Im}(w)-|z|^2\big)+\overline{z}_j z_k}{\big(\mathrm{Im}(w)-|z|^2\big)^2} dz_j\otimes d\overline{z}_k + \frac{dw\otimes d\overline{w}}{4\big(\mathrm{Im}(w)-|z|^2\big)^2} \\
& + \sum_{1\le j\le n-1} \frac{\overline{z}_j dz_j\otimes d\overline{w}}{2i\big(\mathrm{Im}(w)-|z|^2\big)^2} - \sum_{1\le j\le n-1} \frac{z_j dw\otimes d\overline{z}_j}{2i\big(\mathrm{Im}(w)-|z|^2\big)^2}.
\end{aligned}
\tag{3.2}
$$

Similarly, the pull back of $g_{\mathbb{B}^N_{\ell'}}$ by $\Psi^N_{\ell,\ell',n}$ gives a canonical indefinite metric $g_{\mathbb{S}^N_{\ell,\ell',n}} = (\Psi^N_{\ell,\ell',n})^* g_{\mathbb{B}^N_{\ell'}}$ on $\mathbb{S}^N_{\ell,\ell',n}$. A direct computation yields an explicit formula of $g_{\mathbb{S}^N_{\ell,\ell',n}}$ and the corresponding Kähler form $\omega_{\mathbb{S}^N_{\ell,\ell',n}}$. Writing $(Z_1,\cdots,Z_{N-1},W)\in\mathbb{C}^{N-1}\times\mathbb{C}$ for the coordinates of $\mathbb{C}^N$, we have

$$
\omega_{\mathbb{S}^N_{\ell,\ell',n}} = -\sqrt{-1}\partial\bar\partial \log\big(\mathrm{Im}(W)-|Z|^2_{\ell,\ell',n}\big); \tag{3.3}
$$

$$
\begin{aligned}
g_{\mathbb{S}^N_{\ell,\ell',n}} = & \sum_{1\le J,K\le N-1} \frac{\delta_{J,K}\delta_{K,\ell,\ell',n}\big(\mathrm{Im}(W)-|Z|^2_{\ell,\ell',n}\big)+\delta_{J,\ell,\ell',n}\delta_{K,\ell,\ell',n}\overline{Z}_J Z_K}{\big(\mathrm{Im}(W)-|Z|^2_{\ell,\ell',n}\big)^2} dZ_J\otimes d\overline{Z}_K \\
& + \frac{dW\otimes d\overline{W}}{4\big(\mathrm{Im}(W)-|Z|^2_{\ell,\ell',n}\big)^2} + \sum_{1\le J\le N-1} \frac{\delta_{J,\ell,\ell',n}\overline{Z}_J dZ_J\otimes d\overline{W}}{2i\big(\mathrm{Im}(W)-|Z|^2_{\ell,\ell',n}\big)^2} - \sum_{1\le J\le N} \frac{\delta_{J,\ell,\ell',n} Z_J dW\otimes d\overline{Z}_J}{2i\big(\mathrm{Im}(W)-|Z|^2_{\ell,\ell',n}\big)^2}.
\end{aligned}
\tag{3.4}
$$

The CR manifold $\partial\mathbb{B}^n_\ell$ or $\mathbb{H}^n_\ell$ (or $\mathbb{H}^N_{\ell,\ell',n}$) is a fundamental object in CR geometry, serving as the basic model for Levi-nondegenerate hypersurfaces (see [BH]). There are extensive study on the mappings between boundaries of generalized balls. See [BH, BEH, HLTX1, HLTX2, X2, GN] and references therein.

We next recall some preliminary about holomorphic maps between (generalized) Heisenberg hypersurfaces from CR geometry (see [BH], [BEH], [HLTX1]). In this section, we let $F = (\tilde f, g) = (f,\phi,g) = (f_1,\cdots,f_{n-1},\phi_1,\cdots,\phi_{N-n},g)$ be a holomorphic map from a neighborhood U of $p_0\in\mathbb{H}^n_\ell$ into $\mathbb{C}^N$, satisfying $F(U\cap\mathbb{S}^n_\ell)\subset\mathbb{S}^N_{\ell,\ell',n}$ and $F(U\cap\mathbb{H}^n_\ell)\subset\mathbb{H}^N_{\ell,\ell',n}$. We additionally assume $M_1 := U\cap\mathbb{H}^n_\ell$ is connected and F is CR transversal on M_1.

In the following, we denote by $(z,w) = (z_1,\cdots,z_{n-1},w)$ the coordinates of $\mathbb{C}^n = \mathbb{C}^{n-1}\times\mathbb{C}$. Letting $q = (z_0,w_0)\in\mathbb{C}^{n-1}\times\mathbb{C}$ be a point on $\mathbb{H}^n_\ell$, we write $\sigma^0_q(z,w) = (z+z_0, w+w_0+2i\langle z,\bar z_0\rangle_\ell)$ for the (generalized) Heisenberg translation. Then σ^0_q is an automorphism of $\mathbb{S}^n_\ell$, and in particular, σ^0_q is a self-isometry of $(\mathbb{S}^n_\ell, g_{\mathbb{S}^n_\ell})$. We write $\mathrm{Aut}^+(\mathbb{H}^N_{\ell,\ell',n})$ for the group of side-preserving meromorphic automorphisms of $\mathbb{H}^N_{\ell,\ell',n}$. Equivalently, it is the set of rational maps R such that away from the set of indeterminacy I, R locally biholomorphically maps $\mathbb{H}^N_{\ell,\ell',n}$ to itself. In addition, R gives a holomorphic isometric map from $(\mathbb{S}^N_{\ell,\ell',n}\setminus I, g_{\mathbb{S}^N_{\ell,\ell',n}})$ to $(\mathbb{S}^N_{\ell,\ell',n}, g_{\mathbb{S}^N_{\ell,\ell',n}})$.

We will need a normalization lemma from [BH]. To introduce the lemma, we first recall some notations (from [Hu1, Hu2] and [BH]) for functions of weighted degree that will be used in the remaining context of the paper. We parameterize $\mathbb{H}^n_l$ by $(z,\overline{z},u)$ through the map $(z,\overline{z},u)\to(z,u+i\sum_{j=1}^{n-1}\delta_{j,l}|z_j|^2)$. Under the parametrization, we assign the weight of z to be 1, and assign the weight of u and w to be 2. We say a smooth function $h(z,\bar{z},u)$ on $U\cap\mathbb{H}^n_\ell$ is of quantity $O_{wt}(s)$ for $s\in\mathbb{N}$, if $\left|\frac{h(tz,t\bar{z},t^2u)}{t^s}\right|$ is bounded for (z,u) on any compact subset of $U\cap\mathbb{H}^n_l$ and t close to 0. Similarly, we say h is of quantity $o_{wt}(s)$ for $s\in\mathbb{N}$, if $\left|\frac{h(tz,t\bar{z},t^2u)}{t^s}\right|$ converges to 0 uniformly for (z,u) on any compact subset of $U\cap\mathbb{H}^n_\ell$ as t goes to 0. In general, for a smooth function $h(z,\bar{z},u)$ on $U\cap\mathbb{H}^n_\ell$, we denote $h^{(k)}(z,\bar{z},u)$ the sum of terms of weighted degree k in the Taylor expansion of h at 0. Sometimes $h^{(k)}(z,\bar{z},u)$ also denotes a weighted homogeneous polynomial of degree k, if h is not specified. When $h^{(k)}(z,\bar{z},u)$ extends to a holomorphic polynomial of weighted degree k, we write it as $h^{(k)}(z,w)$ or $h^{(k)}(z)$ if it depends only on z.

Lemma 3.1. *Let F be as above. For each $p\in M_1$, there is an element $\beta\in\mathrm{Aut}^+(\mathbb{H}^N_{\ell,\ell',n})$ such that the map $F^{**}_p=\beta\circ F\circ\sigma^0_p$ satisfies the normalization conditions (3.5) and (3.6) when we write $F^{**}_p=(f^{**}_p,\phi^{**}_p,g^{**}_p)$. Here $f^{**}_p=\big((f^{**}_p)_1,\cdots,(f^{**}_p)_{n-1}\big)$ has $(n-1)$ components, ϕ^{**}_p has $N-n$ components and g^{**}_p is a scalar function.*

$$\begin{cases} f^{**}_p = z+\frac{\sqrt{-1}}{2}a^{**(1)}_p(z)w+O_{wt}(4)\\ \phi^{**}_p=\phi^{**(2)}_p(z)+O_{wt}(3)\\ g^{**}_p=w+O_{wt}(5), \end{cases} \tag{3.5}$$

with

$$\langle\bar{z},a^{**(1)}_p(z)\rangle_\ell|z|^2_\ell=|\phi^{**(2)}_p(z)|^2_\tau,\ \tau=\ell'-\ell. \tag{3.6}$$

Remark 3.2. *If we write $a^{**(1)}_p(z)=z\mathcal{A}(p)$ for some $(n-1)\times(n-1)$ matrix $\mathcal{A}(p)=(a_{jk})_{1\le j,k\le n-1}$, then $(f^{**}_p)_k(z,w)=z_k+\frac{\sqrt{-1}}{2}\sum_{j=1}^{n-1}a_{jk}z_jw+O_{wt}(4)$ for $1\le k\le n-1$. By [HLTX1], the geometric rank of F at p is defined as the rank of the matrix $\mathcal{A}(p)$. See more details of the definition in [HLTX1].*

3.2 An algebraic proposition

We first recall the definition of Hermitian rank of a real polynomial. Let $R(z,\overline{z})$ be a real polynomial in $\mathbb{C}^n$. Then $R(z,\overline{z})$ can be written as $R(z,\overline{z}) = \sum_{i=1}^{p} |f_i(z)|^2 - \sum_{j=1}^{q} |g_j(z)|^2$ for some $p, q \in \mathbb{Z}^{\geq 0}$, where f_i's and g_j's are holomorphic polynomials in $\mathbb{C}^n$. Moreover $f_1, \cdots, f_p, g_1, \cdots, g_q$ are linearly independent over $\mathbb{C}$. Then $r = p+q$ is called the (Hermitian) rank of $R(z,\overline{z})$, and the pair (p,q) is called the signature of $R(z,\overline{z})$. We remark that the rank and signature of $R(z,\overline{z})$ are independent of the choices of f_i's and g_j's. The real polynomial $R(z,\overline{z})$ has rank zero (equivalently, it has signature $(0,0)$), if and only if $R(z,\overline{z})$ is identically zero.

We recall the following well-known simple fact about the signature of real polynomials. For the convenience of the readers, we sketch a proof here.

Lemma 3.3. *Let $R(z,\overline{z})$ be a real polynomial. Assume $R(z,\overline{z}) = \sum_{i=1}^{r} |\phi_i(z)|^2 - \sum_{j=1}^{t} |\psi_j(z)|^2$, where $r, t \in \mathbb{Z}^{\geq 0}$; ϕ_i's and ψ_j's are some holomorphic polynomials (they are not necessarily linearly independent over $\mathbb{C}$). Then the signature (p,q) of $R(z,\overline{z})$ satisfies $p \leq r, q \leq t$.*

Proof. We can assume $R(z,\overline{z})$ is not identically zero, for otherwise the conclusion is trivial. Since $R(z,\overline{z})$ has signature (p,q), $R(z,\overline{z})$ can be written as

$$R(z,\overline{z}) = \sum_{i=1}^{p} |f_i(z)|^2 - \sum_{j=1}^{q} |g_j(z)|^2.$$

Here $\{f_1, \cdots, f_p, g_1, \cdots, g_q\}$ is a set of linearly independent holomorphic polynomials over $\mathbb{C}$. We first prove $p \leq r$. Seeking a contradiction, suppose $p > r$. Note by assumption we have

$$\sum_{i=1}^{p} |f_i(z)|^2 + \sum_{j=1}^{t} |\psi_j(z)|^2 = \sum_{i=1}^{r} |\phi_i(z)|^2 + \sum_{j=1}^{q} |g_j(z)|^2.$$

By a lemma of D'Angelo (see [D]), for each $1 \leq i \leq p$, we can find an $(r+q)-$dimensional vector $\mathbf{v_i}$ with complex entries such that $f_i = (\phi_1, \cdots, \phi_r, g_1, \cdots, g_q)\mathbf{v_i}$ for $1 \leq i \leq p$. Consequently, since $p > r$, there exist constants $\lambda_1, \cdots, \lambda_p$, which are not all zero, such that $\sum_{i=1}^{p} \lambda_i f_i$ can be written as a linear combination of g_j's. This contradicts with the linear independence of f_i's and g_j's. Hence we must have $p \leq r$. Similarly one can prove $q \leq t$.

□

We next prove the following algebraic lemma, which will be applied in §3.3. For two m-tuples $\xi = (\xi_1, \cdots, \xi_m)$, $\eta = (\eta_1, \cdots, \eta_m)$ of complex numbers, we write $\langle \xi, \eta \rangle = \sum_{j=1}^{m} \xi_j \eta_j$. Write Tr for the matrix trace operator.

Proposition 3.4. *Let $m \geq 3$ and $z = (z_1, \cdots, z_m)$ be the coordinates of $\mathbb{C}^m$. Let A be an $m \times m$ Hermitian matrix such that $R(z, \overline{z}) = \langle zA, \overline{z} \rangle |z|^2$ has signature (p, q) with $0 \leq q \leq 1$. Then we have the following two conclusions hold.*

(1). $\mathrm{Tr}(A) \geq 0$.

(2). $\mathrm{Tr}(A) = 0$ *if and only if A is the zero matrix.*

Proof of Proposition 3.4: By assumption, we can write

$$R(z, \overline{z}) = \langle zA, \overline{z} \rangle |z|^2 = \sum_{i=1}^{p} |f_i(z)|^2 - \sum_{j=1}^{q} |g_j(z)|^2. \tag{3.7}$$

for some linearly independent holomorphic polynomials $f_1, \cdots, f_p, g_1, \cdots, g_q$. Let U be an $m \times m$ unitary matrix such that $\hat{A} := UA\overline{U}^t = \mathrm{diag}(\lambda_1, \cdots, \lambda_m)$, where $\lambda_1 \geq \cdots \geq \lambda_m$. Replacing z by zU in (3.7), we get

$$\langle z\hat{A}, \overline{z} \rangle |z|^2 = \sum_{i=1}^{p} |f_i(zU)|^2 - \sum_{j=1}^{q} |g_j(zU)|^2. \tag{3.8}$$

Therefore the real polynomial $\hat{R}(z, \overline{z}) := \langle z\hat{A}, \overline{z} \rangle |z|^2$ also has signature (p, q). We next prove the following claim:

Claim 1: One of the following two mutually exlcusive conditions must hold:

(A). For all $1 \leq j \leq m, \lambda_j \geq 0$.

(B). The last eigenvalue $\lambda_m < 0$, and $\lambda_{m-1} \geq -\lambda_m$. Consequently, $\lambda_j > 0$ for every $1 \leq j \leq m-1$ and $\mathrm{Tr}(A) = \sum_{i=1}^{m} \lambda_i > 0$.

Proof of Claim 1: If $\lambda_m \geq 0$, then (A) holds. We will thus assume $\lambda_m < 0$. We restrict $\hat{R}(z, \overline{z})$ to the complex 2−plane $H := \{(0, \cdots, 0, z_{m-1}, z_m) : z_{m-1}, z_m \in \mathbb{C}\}$. Write $\hat{R}|_H$ for the function obtained by this restriction. By the form of $\hat{A}$ and the definition of $\hat{R}$, we have

$$\begin{aligned} \hat{R}|_H &= (\lambda_{m-1}|z_{m-1}|^2 + \lambda_m |z_m|^2)(|z_{m-1}|^2 + |z_m|^2) \\ &= \lambda_{m-1}|z_{m-1}|^4 + (\lambda_{m-1} + \lambda_m)|z_{m-1}z_m|^2 + \lambda_m |z_m|^4. \end{aligned} \tag{3.9}$$

Write (p^*, q^*) for the signature of $\hat{R}|_H$. By restricting (3.8) to H and using Lemma 3.3, we have $q* \leq q \leq 1$.

Suppose $\lambda_{m-1} + \lambda_m < 0$. Note the functions $z_{m-1}^2, z_{m-1}z_m, z_m^2$ are linearly independent over $\mathbb{C}$. Then by (3.9), we have $(p^*, q^*) = (1, 2)$ if $\lambda_{m-1} > 0$; $(p^*, q^*) = (0, 2)$ if $\lambda_{m-1} = 0$;

and $(p^*, q^*) = (0,3)$ if $\lambda_{m-1} < 0$. In any case, it contradicts the preceding conclusion that $q* \leq q \leq 1$. Hence we must have $\lambda_{m-1} \geq -\lambda_m > 0$. Consequently, $\lambda_j > 0$ for all $1 \leq j \leq m-1$. Finally since $m \geq 3$, we have $\sum_{j=1}^m \lambda_j > 0$. This finishes the proof of the claim. ■

We continue to prove Proposition 3.4. Note part (1) of Proposition 3.4 follows immediately from Claim 1. To prove part (2), we only need to show that if $\mathrm{Tr}(A) = 0$, then A is zero. For that, we note when $\mathrm{Tr}(A) = 0$, we must have case (A) holds in Claim 1. In this case, the trace free condition immediately yields that all the eigenvalues $\lambda_j = 0$, and therefore A is the zero matrix. Proposition 3.4 is thus established. ■

Remark 3.5. *In the case $m = 2$, part (1) of Proposition 3.4 still holds, while part (2) fails. For example, write $z = (z_1, z_2)$ for the coordinates of $\mathbb{C}^2$, and let the 2×2 matrix $A = \mathrm{diag}(\lambda, -\lambda)$ for some positive number λ. Then it is clear that the real polynomial $\langle zA, \overline{z} \rangle |z|^2 = \lambda(|z_1|^4 - |z_2|^4)$ has signature $(1,1)$. The trace of A is zero, while A is not zero.*

3.3 Rigidity of isometric maps into products of indefinite hyperbolic spaces

We will prove a rigidity theorem for holomorphic isometric maps into a product of indefinite hyperbolic spaces based on the setup and results from §3.1 and §3.2. The proof uses some recently developed machinery in CR geometry ([HLTX1]), as well as ideas from the work of Yuan-Zhang [YZ].

We consider local holomorphic map sending a piece of $\mathbb{H}_\ell^n$ to $\mathbb{H}_{\ell,\ell',n}^N$ with $\ell = 0$ and $\ell' = 1$. More precisely, let $N > n > 1$ and let $F = (\tilde{f}, g) = (f, \phi, g) = (f_1, \cdots, f_{n-1}, \phi_1, \cdots, \phi_{N-n}, g)$ be a holomorphic map from a neighborhood U of $p_0 = 0 \in \mathbb{H}_0^n$ into $\mathbb{C}^N$, satisfying $F(U \cap \mathbb{S}_0^n) \subset \mathbb{S}_{0,1,n}^N$ and $F(U \cap \mathbb{H}_0^n) \subset \mathbb{H}_{0,1,n}^N$. Assume $F(0) = 0$ and F is CR transversal at 0. Then there exists a positive-valued real analytic function h in a small neighborhood of 0 such that

$$\mathrm{Im}(g) - |\tilde{f}|_{0,1,n}^2 = \big(\mathrm{Im}(w) - |z|^2\big)h.$$

Consequently, $X := g_{\mathbb{S}_0^n} - F^*(g_{\mathbb{S}_{0,1,n}^N})$ extends to a well-defined real analytic Hermitian symmetric $(1,1)-$tensor in some neighborhood V of 0. Similarly as in [YZ], the value of X along $\mathbb{H}_0^n$ gives an intrinsic CR invariant that is associated with the map F near 0. We will follow the idea in [YZ] to make connection of X with the CR second fundamental form of the map F. For that, we normalize F and compute X under the normalization. First note by Lemma 3.1, we can compose F with some $\beta \in \mathrm{Aut}^+(\mathbb{H}_{\ell,\ell',n}^N)$, such that the new map $\beta \circ F$, still denoted by $F = (\tilde{f}, g) = (f, \phi, g)$, satisfies the following normalization (3.10) and (3.11). Here $(z, w) = (z_1, \cdots, z_{n-1}, w)$ denotes the coordinates of $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$.

$$\begin{cases} f = z + \frac{\sqrt{-1}}{2} a^{(1)}(z) w + O_{wt}(4) \\ \phi = \phi^{(2)}(z) + O_{wt}(3) \\ g = w + O_{wt}(5), \end{cases} \tag{3.10}$$

with

$$\langle \bar{z}, a^{(1)}(z) \rangle |z|^2 = |\phi^{(2)}(z)|_1^2. \tag{3.11}$$

Here $a^{(1)}(z) = z\mathcal{A}$ for some $(n-1) \times (n-1)$ matrix $\mathcal{A} = (a_{jk})_{1 \le j,k \le n-1}$. By (3.11), $\mathcal{A}$ is a Hermitian matrix.

Write $H := \mathrm{Im}(g) - |\tilde{f}|^2_{0,1,n}$ which is a real analytic function on U. We next follow the method of Lemma 2.4 and Proposition 2.5 in [YZ] to study the asymptotic behavior of H and the boundary value of X. To make the computation simpler, we will carry it out in a slightly different way from that in [YZ].

Write $w = s + it$, where s, t are real and imaginary parts of w. Write L for a small piece of real line segment passing through 0 along the normal direction $\frac{\partial}{\partial t}$ of $\mathbb{H}^n_0$ at 0. That is, L is the following real line segment for some small $\epsilon > 0$:

$$L = \{(z, w) \in \mathbb{C}^n : z = 0, w = it \text{ with } t \in \mathbb{R}, |t| < \epsilon\}.$$

We restrict H on L to obtain $H|_L$. Since $H(0) = 0$, by the Taylor expansion of $H|_L$ at $t = 0$ we have,

$$H|_L(t) = \frac{\partial H}{\partial t}(0) t + \frac{1}{2} \frac{\partial^2 H}{\partial t^2}(0) t^2 + O(3). \tag{3.12}$$

Here $O(k)$ denotes a real analytic function on L at 0 whose vanishing order at $t = 0$ is at least k. We furthermore have the following lemma.

Lemma 3.6. *Let F be as above and in particular satisfies (3.10) and (3.11). Then $H|_L$ satisfies the following expansion at $t = 0$:*

$$H|_L(t) = t + O(3).$$

Proof of Lemma 3.6: The proof is basically the same as that of Lemma 2.4 in [YZ]. We sketch a proof here. For a function ψ of t, we write $\psi_w = \frac{\partial \psi}{\partial w}$. For a vector-valued function $\Psi = (\psi_1, \cdots, \psi_m)$, we write $\Psi_w = (\frac{\partial \psi_1}{\partial w}, \cdots, \frac{\partial \psi_m}{\partial w})$, and $\psi_{\overline{w}}, \Psi_{\overline{w}}$ are understood similarly. Note $\frac{\partial}{\partial t} = i\frac{\partial}{\partial w} - i\frac{\partial}{\partial \overline{w}}$. We thus have

$$\frac{\partial H}{\partial t}(0) = iH_w - iH_{\overline{w}} = \frac{1}{2}(g_w(0) + \overline{g_w(0)}) + i\langle \tilde{f}(0), \overline{\tilde{f}_w(0)} \rangle_{0,1,n} - i\langle \tilde{f}_w(0), \overline{\tilde{f}}(0) \rangle_{0,1,n} = 1. \tag{3.13}$$

The last equality follows from (3.10). One can verify that $\frac{\partial^2 H}{\partial t^2}(0) = 0$ by using (3.10) in a similar manner. Then the lemma follows from the above calculation and (3.12). ■

Proposition 3.7. *Let F be as Lemma 3.6 and X be as defined above. Write*

$$X = \sum_{1\leq j,k\leq n-1} X_{jk} dz_j \otimes d\overline{z}_k + \sum_{1\leq j\leq n-1} X_{jn} dz_j \otimes d\overline{w} + \sum_{1\leq j\leq n-1} X_{nj} dw \otimes d\overline{z}_j + X_{nn} dw \otimes d\overline{w}.$$

Then we have for $1 \leq j, k \leq n-1$,

$$X_{jk}(0) = -2i \frac{\partial^2 f_k}{\partial z_j \partial w}(0) = a_{jk}.$$

Here $(a_{jk})_{1\leq j,k\leq n-1}$ is the coefficient matrix of $a^{(1)}(z)$ in (3.11).

Proof of Proposition 3.7: The proof is very similar to that of Proposition 2.5 in [YZ]. Although X_{jk} (as well as X_{jn}, X_{nj} and X_{nn}) are real analytic in a neighborhood of 0, we will however carry out a calculation of $H^2 X_{jk}$ instead of X_{jk}, as in [YZ]. This will make the computation easier: In the definition of X, $g_{\mathbb{S}_0^n}$ and $F^*(g_{\mathbb{S}^N_{0,1,n}})$ both have a singularity at 0, and the multiplication of H^2 annihilates the singularity.

Note $H^2 X$ is a real analytic $(1,1)-$tensor near 0 and its coefficient along the direction $dz_j \otimes d\overline{z}_k, 1 \leq j, k \leq n-1$, equals $H^2 X_{jk}$. We restrict $H^2 X_{jk}$ to L, and write h_{jk} for the function obtained by the restriction. Then by Lemma 3.6,

$$h_{jk} = (H|_L)^2 (X_{jk}|_L) = X_{jk}(0) t^2 + O(3). \tag{3.14}$$

On the other hand, we can use the explicit formula $X := g_{\mathbb{S}_0^n} - F^*(g_{\mathbb{S}^N_{0,1,n}})$ to compute $H^2 X$ and h_{jk}. We first consider $H^2 g_{\mathbb{S}_0^n}$. Take the coefficient of $H^2 g_{\mathbb{S}_0^n}$ along $dz_j \otimes d\overline{z}_k, 1 \leq j, k \leq n-1$, and restrict it to L. Denoting by ψ_{jk} the function obtained by this restriction, we have by Lemma 3.6 and (3.2),

$$\psi_{jk} = (H|_L)^2 (\delta_{jk} \frac{t}{t^2}) = \delta_{jk} t + O(3). \tag{3.15}$$

Write $\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}$ for the four tensors on the right hand side of (3.4), respectively. Then we have

$$H^2 F^*(\mathbf{I}) = \sum_{1\leq J,K\leq N-1} \big(\delta_{J,K} \delta_{K,0,1,n} H + \delta_{J,0,1,n} \delta_{K,0,1,n} \overline{\tilde{f}}_J \tilde{f}_K \big) d\tilde{f}_J \otimes d\overline{\tilde{f}_K}.$$

Collect the coefficient of the above tensor along the direction $dz_j \otimes d\overline{z}_k, 1 \leq j, k \leq n-1$ and restrict it to L. Write η^I_{jk} for the function obtained by the restriction. Note by (3.10), $(\tilde{f}_J)|_L(t) = O(2)$ for any $1 \leq J \leq N-1$. Consequently,

$$\eta^I_{jk} = \sum_{1\leq K\leq N-1} \delta_{K,0,1,n} (H|_L) \big(\frac{\partial \tilde{f}_K}{\partial z_j} \big)|_L \big(\frac{\partial \overline{\tilde{f}_K}}{\partial \overline{z}_k} \big)|_L + O(4) = (H|_L) \langle \tilde{f}_{z_j}, \overline{\tilde{f}_{z_k}} \rangle_{0,1,n} + O(4). \tag{3.16}$$

Note $(\phi_l)_{z_j}|_L = O(1)$, for $1 \le l \le N-n, 1 \le j \le n-1$. By the first equation of (3.10), we have

$$(f_l)_{z_j} = \delta_{jl} + \frac{i}{2}a_{jl}w + O_{wt}(3), \quad 1 \le l \le n-1.$$

Using the above and Lemma 3.6, we see (3.16) is reduced to

$$\eta^I_{jk} = t\langle f_{z_j}|_L, \overline{f_{z_k}}|_L\rangle + O(3) = t(\delta_{jk} - \frac{1}{2}\overline{a_{kj}}t - \frac{1}{2}a_{jk}t) + O(3). \tag{3.17}$$

Since $A = (a_{jk})_{1 \le j,k \le n-1}$ is a Hermitian matrix, we have $\overline{a_{kj}} = a_{jk}$. Thus the above equation is reduced to

$$\eta^I_{jk} = \delta_{jk}t - a_{jk}t^2 + O(3). \tag{3.18}$$

Similarly, we collect respectively the coefficient of the tensors $H^2F^*(\mathbf{II}), H^2F^*(\mathbf{III}), H^2F^*(\mathbf{IV})$ along the direction $dz_j \otimes d\overline{z}_k, 1 \le j,k \le n-1$, and restrict them to L. Write the functions obtained by the restriction as $\eta^{II}_{jk}, \eta^{III}_{jk}$ and η^{IV}_{jk}, respectively. Note again by (3.10), $g_{z_j}|_L(t) = O(2), 1 \le j \le n-1$, and $(\tilde{f}_J)|_L(t) = O(2), 1 \le J \le N-1$. One can therefore verify directly that $\eta^{II}_{jk}, \eta^{III}_{jk}$ and η^{IV}_{jk} are all of order $O(4)$. Putting this together with (3.15) and (3.18), we obtain

$$h_{jk} = \psi_{jk} - (\eta^I_{jk} + \eta^{II}_{jk} + \eta^{III}_{jk} + \eta^{IV}_{jk}) = a_{jk}t^2 + O(3). \tag{3.19}$$

Finally we establish Proposition 3.7 by comparing (3.14) and (3.19).

We are now ready to formulate and prove a rigidity result about local holomorphic isometric map from the unit ball to a product of generalized balls.

Theorem 3.8. *Let $n \ge 4$ and $m \ge 1$. Let U be an open subset in $\mathbb{C}^n$ containing some $p \in \partial\mathbb{B}^n$ such that $U \cap \mathbb{B}^n$ is connected. Let $G = (G_1, \cdots, G_m)$ be a holomorphic map from U to $\mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_m}$, where all $N_i \ge 2$. Assume each $G_i, 1 \le i \le m$, satisfies $G_i(U \cap \mathbb{B}^n) \subseteq \mathbb{B}^{N_i}_1$ and $G_i(U \cap \partial\mathbb{B}^n) \subseteq \partial\mathbb{B}^{N_i}_1$. Assume G is a local isometric embedding in the sense that $g_{\mathbb{B}^n} = \sum_{i=1}^m \lambda_i G_i^*(g_{\mathbb{B}^{N_i}_1})$ on $U \cap \mathbb{B}^n$, where λ_i's are all positive constants. Then each G_i is an isometric map from $(U \cap \mathbb{B}^n, g_{\mathbb{B}^n})$ to $(\mathbb{B}^{N_i}_1, g_{\mathbb{B}^{N_i}_1})$ satisfying $G_i^*(g_{\mathbb{B}^{N_i}_1}) = g_{\mathbb{B}^n}$ on $U \cap \mathbb{B}^n$. Consequently, $\sum_{i=1}^m \lambda_i = 1$.*

Remark 3.9. *In the setting of Theorem 3.8, since $1 < n-1$, we can apply Lemma 4.1 in [BH] (or Theorem 1.1 in [BER]) to see that each G_i, as a holomorphic map sending $U \cap \partial\mathbb{B}^n$ to $\partial\mathbb{B}^{N_i}_1$, is CR transversal at a generic point on $\partial\mathbb{B}^n$. It then follows that $N_i \ge n+1$ for each i(cf. [BH]).*

Proof of Theorem 3.8: Recall there is a Cayley transformation Ψ_n that biholomorphically maps $\mathbb{S}_0^n$ and its boundary $\mathbb{H}_0^n$ onto $\mathbb{B}^n$ and $\partial\mathbb{B}^n \setminus \{(0,-1)\}$, respectively. By composing G with some automorphism of $\mathbb{B}^n$, we can assume $p_0 = \Psi_n(0)$. Furthermore, recall there is some generalized Cayley transformation $\Psi_{0,1,n}^N$ that biholomorphically maps $\mathbb{S}_{0,1,n}^N$ and its boundary $\mathbb{H}_{0,1,n}^N$ onto $\mathbb{B}_1^N \setminus \mathcal{V}$ and $\partial\mathbb{B}_1^N \setminus \mathcal{V}$, respectively. Here $\mathcal{V} = \{[z_0, \cdots, z_N] \in \mathbb{P}^N : z_0 + z_N = 0\}$. By composing each G_j with some automorphism of $\mathbb{B}_1^{N_j}$, we can assume $G_i(p_0) = \Psi_{0,1,n}^{N_i}(0)$ for every $1 \leq i \leq m$. Now set

$$F_i := (\Psi_{0,1,n}^{N_i})^{-1} \circ G_i \circ \Psi_n, 1 \leq i \leq m.$$

By the assumption on G_i, each F_i is a well-defined holomorphic map from some neighborhood W of 0 in $\mathbb{C}^n$ to $\mathbb{C}^{N_i}$ with $\Psi_n(W) \subseteq U$ (By shrinking W, we assume $W \cap \mathbb{S}_0^n$ is connected). Moreover, for each i we have, $F_i(W \cap \mathbb{S}_0^n) \subseteq \mathbb{S}_{0,1,n}^{N_i}$ and $F_i(W \cap \mathbb{H}_0^n) \subseteq \mathbb{H}_{0,1,n}^{N_i}$. Furthermore, by the metric preserving condition of G and the definitions of $g_{\mathbb{S}_0^n}$ and $g_{\mathbb{S}_{0,1,n}^{N_i}}$, we have $F := (F_1, \cdots, F_m)$ preserves the metric in the sense that

$$g_{\mathbb{S}_0^n} = \sum_{i=1}^{m} \lambda_i F_i^*(g_{\mathbb{S}_{0,1,n}^{N_i}}) \text{ on } W \cap \mathbb{S}_0^n. \tag{3.20}$$

The argument in Remark 3.9 shows that F_i, as a holomorphic map sending $W \cap \mathbb{H}_0^n$ to $\mathbb{H}_{0,1,n}^{N_i}$, is CR transversal at a generic point on $W \cap \mathbb{H}_0^n$. By shrinking W to a small ball centered at some $q \in \mathbb{H}_0^n$, we can assume F_i is CR transversal along $W \cap \mathbb{H}_0^n$ for every $1 \leq i \leq m$. By composing F with σ_q^0, we can further assume $q = 0$. Furthermore, by Lemma 3.1, composing each F_i with some element in $\mathrm{Aut}^+(\mathbb{H}_{0,1,n}^{N_i})$ if necessary, we can assume F_i satisfies the normalization condition (3.10) and (3.11). Following the notation at the beginning of this section, we write $X(F_i) := g_{\mathbb{S}_0^n} - F_i^*(g_{\mathbb{S}_{0,1,n}^{N_i}})$. By the previous discussion, $X(F_i)$ extends to a real analytic $(1,1)-$tensor in some small neighborhood of 0, which can be indeed taken to be W. We rewrite (3.20) into the following equation:

$$(-1 + \sum_{i=1}^{m} \lambda_i) g_{\mathbb{S}_0^n} = \sum_{i=1}^{m} \lambda_i (g_{\mathbb{S}_0^n} - F_i^*(g_{\mathbb{S}_{0,1,n}^{N_i}})) = \sum_{i=1}^{m} \lambda_i X(F_i) \text{ on } W \cap \mathbb{S}_0^n. \tag{3.21}$$

Note the right hand side of (3.21) extends real analytically across $W \cap \mathbb{H}_0^n$, while $g_{\mathbb{S}_0^n}$ is singular at every point on $\mathbb{H}_0^n$. Hence we must have $\sum_{i=1}^m \lambda_i = 1$. Consequently, (3.21) is reduced to

$$\sum_{i=1}^{m} \lambda_i X(F_i) = 0 \text{ on } W \cap \mathbb{S}_0^n. \tag{3.22}$$

Collecting the $(dz_j \otimes d\overline{z}_k)-$component of (3.22), $1 \le j, k \le n-1$, and then letting $z \in W \cap \mathbb{S}_0^n$ go to 0, we obtain

$$\sum_{i=1}^{m} \lambda_i \big(X(F_i)\big)_{jk}(0) = 0. \tag{3.23}$$

Here $\big(X(F_i)\big)_{jk}$ is $(dz_j \otimes d\overline{z}_k)-$component of $X(F_i)$. By Proposition 3.7, for each $1 \le i \le m$, $\big(X(F_i)\big)_{jk}(0) = a^i_{jk}$, where $\mathcal{A}^i = (a^i_{jk})_{1 \le j,k \le n-1}$ is the matrix associated with F_i in the expansion of F_i at 0 as in (3.10) and (3.11) (see also Lemma 3.1 and Remark 3.2). Recall by (3.11), $\mathcal{A}^i$ is a Hermitian matrix. Moreover, by Lemma 3.3 and (3.11), $\langle z\mathcal{A}^i, \overline{z}\rangle |z|^2$ has signature (p, q) with $0 \le q \le 1$. Since $n-1 \ge 3$, by part (1) of Proposition 3.4, $\mathrm{Tr}(\mathcal{A}^i) \ge 0$. But (3.23) means $\sum_{i=1}^m \lambda_i \mathcal{A}^i = 0$, which implies $\sum_{i=1}^m \lambda_i \mathrm{Tr}(\mathcal{A}^i) = 0$. Hence we must have $\mathrm{Tr}(\mathcal{A}^i) = 0$ for each i. We then apply part (2) of Proposition 3.4 to conclude that $\mathcal{A}^i = 0$ for every i. It follows that the geometric rank of each F_i at 0 equals 0 (see Remark 3.2).

Next for each $p \in W \cap \mathbb{H}_0^n$ near 0, let $\beta_i \in \mathrm{Aut}^+(\mathbb{H}^{N_i}_{0,1,n})$ (depending on p) be such that $(F_i)^{**}_p := \beta_i \circ F_i \circ \sigma^0_p, 1 \le i \le m$, satisfies the normalization as in Lemma 3.1 with $\ell = 0, \ell' = 1$ (or the normalization similar to (3.10) and (3.11)). In particular, each $(F_i)^{**}_p$ maps 0 to 0. Moreover, $\hat{F}_p := ((F_1)^{**}_p, \cdots, (F_m)^{**}_p)$ still satisfies the metric preserving condition as in (3.20). Then we can apply the preceding argument to $\hat{F}_p$ and conclude the geometric rank of each $(F_i)^{**}_p$ at 0 equals 0. This is equivalent to that F_i has the geometric rank 0 at p (see Proposition 3.4 in [HLTX1]). Since p is an arbitrary point on $\mathbb{H}_0^n$ near 0, it follows that each F_i has geometric rank 0 at every point on $\mathbb{H}_0^n$ near 0. By the relation of F_i and G_i, and the definition of geometric rank (see page 14 [HLTX1]), we see each G_i is CR transversal and has geometric rank zero along some open piece of $\partial\mathbb{B}^n \cap U$. Finally we apply Theorem 1 in [HLTX1] to conclude that every G_i is an isometric map from $(U \cap \mathbb{B}^n, g_{\mathbb{B}^n})$ to $(\mathbb{B}_1^{N_i}, g_{\mathbb{B}_1^{N_i}})$ with $G_i^*(g_{\mathbb{B}_1^{N_i}}) = g_{\mathbb{B}^n}$ on $U \cap \mathbb{B}^n$. Theorem 3.8 is thus established. ■

Corollary 3.10. *Let $n \ge 4$ and $m \ge 1$. Let U be an open subset in $\mathbb{C}^n$ containing some $p \in \partial\mathbb{B}^n$ such that $U \cap \mathbb{B}^n$ is connected. Let $G = (G_1, \cdots, G_m)$ be a holomorphic map from U to $\mathbb{P}^{M_1} \times \cdots \times \mathbb{P}^{M_m}$, where all $M_i \ge 2$. Let $D_i \subseteq \mathbb{P}^{M_i}, 1 \le i \le m$, be either the unit ball $\mathbb{B}^{M_i} \subset \mathbb{C}^{M_i} \subset \mathbb{P}^{M_i}$ or the generalized ball $\mathbb{B}_1^{M_i} \subset \mathbb{P}^{M_i}$. Assume each $G_i, 1 \le i \le m$, satisfies $G_i(U \cap \mathbb{B}^n) \subseteq D_i$ and $G_i(U \cap \partial\mathbb{B}^n) \subseteq \partial D_i$. Assume G is a local isometric embedding in the sense that $g_{\mathbb{B}^n} = \sum_{i=1}^m \lambda_i G_i^*(g_{D_i})$ on $U \cap \mathbb{B}^n$, where $\lambda_i' s$ are all positive constants. Then each F_i is an isometric map from $(U \cap \mathbb{B}^n, g_{\mathbb{B}^n})$ to (D_i, g_{D_i}) satisfying $G_i^*(g_{D_i}) = g_{\mathbb{B}^n}$ on $U \cap \mathbb{B}^n$. Consequently, $\sum_{i=1}^m \lambda_i = 1$.*

Proof of Corollary 3.10: Write T_M for the standard embedding from $\mathbb{C}^M$ to $\mathbb{P}^{M+1}$ given by

$$T_M : (z_1, \cdots, z_M) \in \mathbb{C}^M \to [1, 0, z_1, \cdots, z_M] \in \mathbb{P}^{M+1}.$$

Note T_M gives a canonical holomorphic isometric embedding from $(\mathbb{B}^M, g_{\mathbb{B}^M})$ to $(\mathbb{B}_1^{M+1}, g_{\mathbb{B}_1^{M+1}})$: $T_M^*(g_{\mathbb{B}_1^{M+1}}) = g_{\mathbb{B}^M}$. In particular, T_M maps $\partial\mathbb{B}^M$ to $\partial\mathbb{B}_1^{M+1}$.

We define a new map $\widetilde{G} = (\widetilde{G}_1, \cdots, \widetilde{G}_m)$ in terms of G as follows. For $1 \leq i \leq m$, if D_i is the generalized ball $\mathbb{B}_1^{M_i}$, then we set $N_i = M_i$ and $\widetilde{D}_i = D_i$, and set $\widetilde{G}_i = G_i$. If D_i is the unit ball $\mathbb{B}^{M_i}$, then we set $N_i = M_i + 1$ and $\widetilde{D}_i = \mathbb{B}_1^{N_i}$, and set $\widetilde{G}_i = T_{M_i} \circ G_i : U \to \mathbb{P}^{N_i}$. Then $\widetilde{G}$ is a holomorphic map from U to $\mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_m}$ satisfying $\widetilde{G}_i(U \cap \mathbb{B}^n) \subseteq \mathbb{B}_1^{N_i}$ and $\widetilde{G}_i(U \cap \partial\mathbb{B}^n) \subseteq \partial\mathbb{B}_1^{N_i}$. Moreover, $\widetilde{G}$ is isometric in the sense that $g_{\mathbb{B}^n} = \sum_{i=1}^m \lambda_i \widetilde{G}_i^*(g_{\mathbb{B}_1^{N_i}})$. Hence by Theorem 3.8, each $\widetilde{G}_i$ is an isometric map with $g_{\mathbb{B}^n} = \widetilde{G}_i^*(g_{\mathbb{B}_1^{N_i}})$. Note in the case $D_i = \mathbb{B}^{M_i}$, we have

$$g_{\mathbb{B}^n} = \widetilde{G}_i^*(g_{\mathbb{B}_1^{N_i}}) = G_i^*\big(T_M^*(g_{\mathbb{B}_1^{N_i}})\big) = G_i^*(g_{\mathbb{B}^{M_i}}) \text{ on } U \cap \mathbb{B}^n.$$

This proves Corollary 3.10. ■

Remark 3.11. *In the setting of Corollary 3.10, from the proof and Remark 3.9, we see for each* $1 \leq i \leq m$, *if* $D_i = \mathbb{B}^{M_i}$, *then we must have* $M_i \geq n$. *If* $D_i = \mathbb{B}_1^{M_i}$, *then we must have* $M_i \geq n + 1$.

4 Proof of Theorem 1.3

In this section, we give a proof of Theorem 1.3. Let F be as in the theorem. First, as discussed in Section 1, by [M3] and [CXY], F extends to a holomorphic proper and isometric immersion from $\mathbb{B}^n$ to Ω. We can thus just assume $V = \mathbb{B}^n$. Before starting the proof, we remark that in the setting of Theorem 1.3, D_2^{IV} cannot appear as one of the Ω_i's. Indeed, suppose $\Omega_i = D_2^{IV}$ for some i. Since D_2^{IV} is biholomorphic to the bidisc Δ^2, it follows from Theorem 1.1 that a generic point on $\partial\mathbb{B}^n$ is mapped to the unit circle $\partial\Delta$ by some non-constant holomorphic map. This is a contradiction since $n \geq 4$ (cf. [BX]). Hence if $\Omega_i = D_{N_i}^{IV}$, then we must have $N_i \geq 3$. Consequently, each Ω_i is an irreducible bounded symmetric domain. We also remark that once Theorem 1.3 is established, then by the existence of an isometric map from $\mathbb{B}^n$ to Ω_i we must have $N_i \geq n$ when $\Omega_i = \mathbb{B}^{N_i}$; and $N_i \geq n + 1$ when $\Omega_i = D_{N_i}^{IV}$(see [M4] or [XY1]. The same conclusion can also be derived by merely applying Theorem 1.1 which yields the existence of a non-constant holomorphic map sending a piece of $\partial\mathbb{B}^n$ to $\partial\Omega_i$).

Proof of Theorem 1.3: Denote by L_N the following embedding from $\mathbb{C}^N$ to $\mathbb{P}^{N+1}$:

$$L_N(z_1, \cdots, z_N) = [1, \frac{1}{2}\sum_{j=1}^N z_j^2, z_1, \cdots, z_N].$$

Then by (1.2) and (3.1), L_N gives a canonical holomorphic isometric map from $(D_N^{IV}, g_{D_N^{IV}})$ to $(\mathbb{B}_1^{N+1}, g_{\mathbb{B}_1^{N+1}}) : L_N^*(g_{\mathbb{B}_1^{N+1}}) = g_{D_N^{IV}}$. In particular, we have $L_N(\partial D_N^{IV}) \subseteq \partial \mathbb{B}_1^{N+1}$. Note by Theorem 1.1, there exists a small ball U centered at some $p_0 \in \partial \mathbb{B}^n$ (in particular $U \cap \mathbb{B}^n$ is connected) such that F extends holomorphically to U. Moreover, still denoting the extension by F, for each $1 \leq i \leq m, F(U \cap \partial \mathbb{B}^n) \subseteq \partial \Omega_i$ (and trivially $F(U \cap \mathbb{B}^n) \subseteq \Omega_i$). Then we define a new map $G = (G_1, \cdots, G_m)$ on U in terms of F as follows. For each $1 \leq i \leq m$, if $\Omega_i = \mathbb{B}^{N_i}$, then we just define $M_i = N_i$ and $D_i = \mathbb{B}^{M_i}$, and define $G_i = F_i$ on U. If $\Omega_i = D_{N_i}^{IV}$, then we define $M_i = N_i + 1$ and $D_i = \mathbb{B}_1^{M_i}$, and define $G_i = L_{N_i} \circ F_i$ on U. Here L_{N_i} is the aforementioned embedding from $\mathbb{C}^{N_i}$ to $\mathbb{P}^{M_i}$ that gives a canonical holomorphic isometric map from $(D_{N_i}^{IV}, g_{D_{N_i}^{IV}})$ to $(\mathbb{B}_1^{M_i}, g_{\mathbb{B}_1^{M_i}})$. One can easily verify that G_i maps U to $\mathbb{P}^{M_i}$, and satisfies $G_i(U \cap \mathbb{B}^n) \subseteq D_i$ and $G_i(U \cap \partial \mathbb{B}^n) \subseteq \partial D_i$. In this way, $G = (G_1, \cdots, G_m)$ is a local holomorphic map from U to $\mathbb{P}^{M_1} \times \cdots \times \mathbb{P}^{M_m}$. Moreover, by the definition of G as well as the metric-preserving property of F and L_{N_i}, we have $g_{\mathbb{B}^n} = \sum_{i=1}^m \lambda_i G_i^*(g_{D_i})$ on $U \cap \mathbb{B}^n$. Consequently, G satisfies the assumptions in Corollary 3.10. Hence by the conclusion of Corollary 3.10, each $G_i, 1 \leq i \leq m$, is an isometric map from $U \cap \mathbb{B}^n$ to $D_i : g_{\mathbb{B}^n} = G_i^*(g_{D_i})$ on $U \cap \mathbb{B}^n$.

Now for each i, by the construction of G, if D_i equals $\mathbb{B}^{N_i}$, then so does Ω_i and $F_i = G_i$. Therefore F_i is a local holomorphic isometric map from $U \cap \mathbb{B}^n$ to $\Omega_i = \mathbb{B}^{N_i}$, and thus extends to a totally geodesic embedding to $\mathbb{B}^n$ to $\mathbb{B}^{N_i}$. If $D_i = \mathbb{B}_1^{M_i}$, then $\Omega_i = D_{N_i}^{IV}$ with $M_i = N_i + 1$, and $G_i = L_{N_i} \circ F_i$. It then follows that

$$g_{\mathbb{B}^n} = (L_{N_i} \circ F_i)^*(g_{\mathbb{B}_1^{N_i+1}}) = F_i^*\big(L_{N_i}^*(g_{\mathbb{B}_1^{N_i+1}})\big) = F_i^*(g_{D_{N_i}^{IV}}) \text{ on } U \cap \mathbb{B}^n.$$

Hence for every $1 \leq i \leq m$, F_i is a local holomorphic isometric map from $(U \cap \mathbb{B}^n, g_{\mathbb{B}^n})$ to (Ω_i, g_{Ω_i}). Finally since F is holomorphic on $\mathbb{B}^n$ (see the discussion at the beginning of this section), by the analyticity we see every F_i is holomorphic isometric map from $\mathbb{B}^n$ to Ω_i. This finishes the proof of Theorem 1.3. ■

We finally remark that, in the proofs of Theorem 3.8 and Theorem 1.3, the assumption $n \geq 4$ is essentially used to apply Proposition 3.4, which yields the vanishing of the geometric rank. When $n = 2, 3$, it seems the behavior of the geometric rank can be more complicated and in particular, Proposition 3.4 fails when $m = 1, 2$ (see Remark 3.5). We however expect the conclusion of Theorem 1.3 to still hold in these lower dimensional cases.

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M. Xiao, Department of Mathematics, University of California, San Diego, 9500 Gilman Drive, La Jolla, CA 92093, USA. (m3xiao@ucsd.edu)