Proper mappings between indefinite hyperbolic spaces and type I classical domains

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Abstract

In this paper, we first study a mapping problem between indefinite hyperbolic spaces by employing the work established earlier by the authors. In particular, we generalize certain theorems proved by Baouendi-Ebenfelt-Huang and Ng. Then we use these results to prove a rigidity result for proper holomorphic mappings between type I classical domains, which confirms a conjecture formulated by Chan after the work of Zaitsev-Kim, Kim and himself.

1 Introduction

The purpose of the paper is twofold. The first part of the paper establishes rigidity results for proper holomorphic maps between indefinite hyperbolic spaces. The second part of the paper is devoted to studying rigidity problem of proper maps between classical domains by using results in the first part and the machinery established by Ng and Chan. To present our main results, we first recall some standard notions in literature. Given integers $n \geq 2$

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and $0 \leq l \leq n-1$, the generalized complex unit ball is defined as the following domain in $\mathbb{P}^n$:

$$\mathbb{B}_l^n = \{[z_0, ..., z_n] \in \mathbb{P}^n : |z_0|^2 + ... + |z_l|^2 > |z_{l+1}|^2 + ... + |z_n|^2\}.$$

For $0 \leq k \leq m$, let $I_{k,m}$ be the $m \times m$ diagonal matrix, where its first k diagonal elements equal -1 and the rest equal 1. Denote by $SU(l+1, n+1)$ the special indefinite unitary group that consists of matrices $A \in SL(n+1, \mathbb{C})$ satisfying $AI_{l+1,n+1}\overline{A}^t = I_{l+1,n+1}$. Then the generalized ball $\mathbb{B}_l^n$ is indeed an open orbit of the real form $SU(l+1, n+1)$ of the complex simple Lie group $SL(n+1, \mathbb{C})$ when acting on $\mathbb{P}^n$. The generalized ball $\mathbb{B}_l^n$ possesses a canonical indefinite metric $\omega_{\mathbb{B}_l^n}$ that is invariant under the action of its automorphism group $SU(l+1, n+1)$:

$$\omega_{\mathbb{B}_l^n} = -\sqrt{-1}\partial\bar{\partial}\log(\sum_{j=0}^{l} |z_j|^2 - \sum_{j=l+1}^{n} |z_j|^2).$$

The generalized ball equipped with the above indefinite metric is often called an indefinite hyperbolic space form. It is reduced to the standard hyperbolic space form (up to a normalization of metric) in the special case $l = 0$.

Under the action of $SU(l+1, n+1)$ on $\mathbb{P}^n$, the topological boundary $\partial\mathbb{B}_l^n$ of $\mathbb{B}_l^n$, is the unique closed orbit. It is often called the generalized sphere of signature l. Much attention has been paid to the study of holomorphic mappings between generalized spheres. And striking rigidity phenomena have been discovered due to the distinct CR geometric structure of the generalized spheres. In this paper, we will concentrate on the case $l > 0$ (The readers are referred to [Hu1, Hu2, HJ, DX, NTY] and references therein for the case $l = 0$). Local holomorphic mappings that send an open piece of $\partial\mathbb{B}_l^n$ into a higher dimensional generalized sphere $\partial\mathbb{B}_{l'}^N$ with $l > 0$ were first studied by Baouendi-Huang [BH] and Baouendi-Ebenfelt-Huang [BEH] . We recall the following rigidity result from [BEH].

Theorem 0.1 (Baouendi-Ebenfelt-Huang [BEH]) Let $N \geq n$, $1 \leq l \leq \frac{n-1}{2}$, $1 \leq l' \leq \frac{N-1}{2}$ and $1 \leq l \leq l' < 2l$. Let U be an open subset in $\mathbb{P}^n$ containing some $p \in \partial\mathbb{B}_l^n$ with $U \cap \mathbb{B}_l^n$ being connected, and F a holomorphic map from U into $\mathbb{P}^N$. Assume $F(U \cap \mathbb{B}_l^n) \subseteq \mathbb{B}_{l'}^N$ and $F(U \cap \partial\mathbb{B}_l^n) \subseteq \partial\mathbb{B}_{l'}^N$. Then F is an isometric embedding from $(U \cap \mathbb{B}_l^n, \omega_{\mathbb{B}_l^n})$ into $(\mathbb{B}_{l'}^N, \omega_{\mathbb{B}_{l'}^N})$.

Here we say F is isometric if it preserves the indefinite hyperbolic metrics: $F^*(\omega_{\mathbb{B}_{l'}^N}) = \omega_{\mathbb{B}_l^n}$ on $U \cap \mathbb{B}_l^n$. By using a different approach from [BEH] that utilizes structure of the moduli space of linear subspaces contained in generalized balls, Ng establishes the global version of Theorem 0.1.

Theorem 0.2 (Ng [Ng1]) Let $1 \leq l < \frac{n}{2}, 1 \leq l' < \frac{N}{2}$ and $f : \mathbb{B}_l^n \to \mathbb{B}_{l'}^N$ be a proper holomorphic map. If $l' \leq 2l - 1$, then f extends to a linear embedding of $\mathbb{P}^n$ into $\mathbb{P}^N$.

In a recent paper [HLTX], the authors gave a complete characterization for local holomorphic isometric embeddings between indefinite hyperbolic spaces in terms of a boundary invariant of the maps—their geometric rank (see [HLTX] for more details).

In the first part of this paper, we further investigate holomorphic maps between generalized balls by making use of the characterization established in [HLTX]. We prove the following Theorem 1.1 and Corollary 1.3, that generalize Theorem 0.1 and Theorem 0.2, respectively.

Theorem 1.1. *Let $N \geq n \geq 3$, $1 \leq l \leq n-2, l \leq l' \leq N-1$. Let U be an open subset in $\mathbb{P}^n$ containing some $p \in \partial \mathbb{B}_l^n$ and F be a holomorphic map from U into $\mathbb{P}^N$. Assume $U \cap \mathbb{B}_l^n$ is connected and $F(U \cap \mathbb{B}_l^n) \subseteq \mathbb{B}_{l'}^N$, $F(U \cap \partial \mathbb{B}_l^n) \subseteq \partial \mathbb{B}_{l'}^N$. Assume one of the following conditions holds:*

$$
\begin{aligned}
&(1).\quad l' < 2l, l' < n-1;\\
&(2).\quad l' < 2l, N-l' < n;\\
&(3).\quad N-l' < 2n-2l-1, l' < n-1;\\
&(4).\quad N-l' < 2n-2l-1, N-l' < n.
\end{aligned}
$$

Then F is an isometric embedding from $(U \cap \mathbb{B}_l^n, \omega_{\mathbb{B}_l^n})$ to $(\mathbb{B}_{l'}^N, \omega_{\mathbb{B}_{l'}^N})$.

Note Theorem 1.1 implies Theorem 0.1 as a special case. Indeed, the assumption in Theorem 0.1 yields that the condition in (1) in Theorem 1.1 holds. We now pause to introduce the following definition.

Definition 1.2. *Let F be a holomorphic rational map from $\mathbb{P}^n$ to $\mathbb{P}^N$. Write $I \subseteq \mathbb{P}^n$ for the set of indeterminacy of F. We say F is a rational proper map from $\mathbb{B}_l^n$ to $\mathbb{B}_{l'}^N$, if F maps from $\mathbb{B}_l^n \setminus I$ to $\mathbb{B}_{l'}^N$ and maps $\partial \mathbb{B}_l^n \setminus I$ to $\partial \mathbb{B}_{l'}^N$.*

Theorem 1.1 can be immediately applied to study rational proper maps between generalized balls.

Corollary 1.3. *Let $N \geq n \geq 3$, $1 \leq l \leq n-2, l \leq l' \leq N-1$. Assume one of the conditions in (1)–(4) of Theorem 1.1 holds. Let F be a rational proper map from $\mathbb{B}_l^n$ to $\mathbb{B}_{l'}^N$. Then F is a linear embedding from $\mathbb{P}^n$ to $\mathbb{P}^N$. Moreover, there exists $h \in Aut(\mathbb{B}_{l'}^N)$ such that*

$$h \circ F([z]) = [z_0, ..., z_l, 0, ..., 0, z_{l+1}, ..., z_n, 0, ..., 0],$$

for $[z] = [z_0, ..., z_l, z_{l+1}, ..., z_n] \in \mathbb{P}^n$, where the first zero tuple has $l' - l$ components.

Remark 1.4. *Note if $l \geq 1$, then every proper holomorphic map from $\mathbb{B}^n_l$ to $\mathbb{B}^N_{l'}$ extends to a rational map from $\mathbb{P}^n$ to $\mathbb{P}^N$ (see [Ng1]). Thus Corollary 1.3 still holds if we assume F is a proper holomorphic from $\mathbb{B}^n_l$ to $\mathbb{B}^N_{l'}$ instead of assuming it is a rational proper map from $\mathbb{B}^n_l$ to $\mathbb{B}^N_{l'}$. Hence Corollary 1.3 has Theorem 0.2 as its special case. (Notice that the assumption in Theorem 0.2 yields that Condition (1) holds). It also has Corollary 1.6 in [BEH] as its special case. (One verifies that the condition in (1) or (4) holds in the setting of Corollary 1.6 in [BEH]).*

The following remark shows that Theorem 1.1 and Corollary 1.3 are optimal in a certain sense.

Remark 1.5. *Theorem 1.1 is optimal in the sense that it fails if none of the conditions (1)–(4) holds. Indeed, suppose all of the conditions (1)–(4) fail. Then one of the following two cases must hold: (A). $l' \geq 2l$ and $N-l' \geq 2n-2l-1$; (B). $N-l' \geq n$ and $l' \geq n-1$. The next two examples show the conclusion in Theorem 1.1 fails in each of the cases. Example 1.6 corresponds to the case (A) with $l' = 2l$ and $N-l' = 2n-2l-1$. Example 1.7 corresponds to the case (B) with $N-l' = n$ and $l' = n-1$. Furthermore, the map in Example 1.6 is indeed a rational proper map between the generalized balls in the sense of Definition 1.2. Thus it also shows Corollary 1.3 fails if none of the conditions (1)–(4) holds.*

Example 1.6. *(Generalized Whitney map from $\mathbb{B}^{l+k}_l$ to $\mathbb{B}^{2l+2k-1}_{2l}$) Let $l \geq 1, k \geq 1$. Write $[w,z] = [w_0, w_1, \cdots, w_l, z_1, \cdots, z_k]$ for the homogeneous coordinates of $\mathbb{P}^{l+k}$ and*

$$\mathbb{B}^{l+k}_l = \Big\{[w,z] \in \mathbb{P}^{k+l} : \sum_{i=0}^{l} |w_i|^2 > \sum_{j=1}^{k} |z_j|^2\Big\}.$$

Write $U = \mathbb{P}^{k+l} \setminus \{w_0 = z_k = 0\}$. Consider the following map $G : U \to \mathbb{P}^{2k+2l-1}$:

$$\begin{aligned} G([w,z]) = [&w_0^2, w_0w_1, \cdots, w_0w_l, w_1z_k, w_2z_k, \cdots, w_lz_k, \\ &w_0z_1, w_0z_2, \cdots, w_0z_{k-1}, z_1z_k, z_2z_k, \cdots, z_{k-1}z_k, z_k^2]. \end{aligned}$$

Write the above components on the right hand side as $G_1, \cdots, G_{2k+2l}$ and set $|G|^2_{2l+1} = -\sum_{i=1}^{2l+1} |G_i|^2 + \sum_{j=2l+2}^{2k+2l} |G_j|^2$. Notice that $|G|^2_{2l+1} = (|w_0|^2 + |z_k|^2)(-\sum_{i=0}^{l} |w_i|^2 + \sum_{j=1}^{k} |z_j|^2)$. Consequently, G maps $U \cap \mathbb{B}^{l+k}_l$ to $\mathbb{B}^{2l+2k-1}_{2l}$ and maps $U \cap \partial\mathbb{B}^{l+k}_l$ to $\partial\mathbb{B}^{2l+2k-1}_{2l}$. Hence the statement in Theorem 1.1 fails in this case.

Furthermore, the set of indeterminacy of G is given by $\{w_0 = z_k = 0\}$, and G is a rational proper map from $\mathbb{B}^{l+k}_l$ to $\mathbb{B}^{2l+2k-1}_{2l}$ in the sense of Definition 1.2. Therefore it shows Corollary 1.3 fails in the case (A). When $l = 1$ and $k = 2$, it also gives a counterexample for Corollary 1.3 in the case (B).

Example 1.7. *(Generalized Whitney map from $\mathbb{B}_l^{l+k}$ to $\mathbb{B}_{l+k-1}^{2l+2k-1}$) Let $l \geq 1, k \geq 1$. Let the homogeneous coordinates $[w, z]$ and $\mathbb{B}_l^{l+k} \subseteq \mathbb{P}^{l+k}$ be the same as in Example 1.6. Let $V = \mathbb{P}^{l+k} \setminus \{w_0 = w_l = 0\}$ and $H : V \to \mathbb{P}^{2k+2l-1}$ be defined as follows:*

$$H([w,z]) = [w_0^2, w_0w_1, \cdots w_0w_{l-1}, w_lz_1, w_lz_2, \cdots, w_lz_k, \\ w_0z_1, w_0z_2, \cdots, w_0z_k, w_1w_l, w_2w_l, \cdots, w_l^2].$$

Write the above components on the right hand side as $H_1, \cdots, H_{2k+2l}$ and set $|H|^2_{l+k} = -\sum_{i=1}^{l+k} |H_i|^2 + \sum_{j=l+k+1}^{2k+2l} |H_j|^2$. Notice that $|H|^2_{l+k} = (|w_0|^2 - |w_l|^2)(-\sum_{i=0}^{l} |w_i|^2 + \sum_{j=1}^{k} |z_j|^2)$. Thus H maps $V \cap \partial\mathbb{B}_l^{l+k}$ to $\partial\mathbb{B}_{l+k-1}^{2l+2k-1}$. In particular, set $V_+ := \{[w,z] \in V : |w_0| > |w_l|\}$. Then H maps $V_+ \cap \mathbb{B}_l^{l+k}$ to $\mathbb{B}_{l+k-1}^{2l+2k-1}$ and maps $V_+ \cap \partial\mathbb{B}_l^{l+k}$ to $\partial\mathbb{B}_{l+k-1}^{2l+2k-1}$. Hence the statement in Theorem 1.1 fails in this case. This map H is, however, not a rational proper map from $\mathbb{B}_l^{l+k}$ to $\mathbb{B}_{l+k-1}^{2l+2k-1}$ in the sense of Definition 1.2, as it maps some point in $\mathbb{B}_l^{l+k}$ to $\mathbb{P}^{2l+2k-1} \setminus \mathbb{B}_{l+k-1}^{2l+2k-1}$.

In the second part of the paper, we apply Theorem 1.1 and Corollary 1.3 to study a mapping problem between type I classical domains. The study of proper holomorphic maps between bounded symmetric domains of high rank goes back to the work of Tumanov-Henkin [TH] (see also Henkin-Novikov [HN]). They proved that any proper self-mapping of an irreducible bounded symmetric domain of rank at least two is an automorphism. Since then, rigidity and classification problems for holomorphic proper maps between bounded symmetric domains have attracted much attention. Let $F : \Omega_1 \to \Omega_2$ be a proper holomorphic map between two bounded symmetric domains Ω_1 and Ω_2. Tsai [Ts] proved the total geodesy of F under the assumption that $\mathrm{rank}(\Omega_1) \geq \mathrm{rank}(\Omega_2) \geq 2$ and Ω_1 is irreducible. Much less is known about the remaining case when $\mathrm{rank}(\Omega_1) < \mathrm{rank}(\Omega_2)$ and the studies so far are mainly focused on the type I classical domains. Many interesting results along these lines can be found in [M1, T1, T2, Ng2, KZ1, KZ2, K, S1, S2, Ch]. We pause to recall the definition of the type I classical domains. Let r and s be positive integers. Write $M(r, s; \mathbb{C})$ for the set of all $r \times s$ complex matrices and I_s for the $s \times s$ identity matrix. The type I classical domain $D^I_{r,s}$ is defined by

$$D^I_{r,s} = \{Z \in M(r,s;\mathbb{C}) : I_s - \overline{Z}^t Z > 0\}.$$

An important step toward understanding proper maps between type I classical domains was due to Ng [Ng2] where he first found its deep connection with mapping problems for proper maps between generalized balls. In such a way, he would be able to apply results in CR geometry to mapping problems between bounded symmetric domains. Among other things, he proved that every proper holomorphic map $f : D^I_{r,s} \to D^I_{r',s}$ is standard (i.e., totally geodesically isometric embedding, up to normalization constants, with respect to the

Bergman metrics) if $s \geq r \geq 2$ and $r' \leq \min\{2r-1, s\}$. In a recent nice paper of Chan [Ch], he posed the following conjecture, that was inspired by the work of Kim-Zaitsev [KZ2], Kim [K] and his own investigation in [Ch]. The statement in the conjecture would generalize the aforementioned theorem of Ng when $r' < s$.

Conjecture 1.8. *Let $f : D^I_{p,q} \to D^I_{p',q'}, p \geq q > 1$ be a proper holomorphic map. Assume $q' < p$ and one of the following conditions holds: (1) $p' < 2p-1$; (2) $q' < 2q-1$. Then*
(I). $p' \geq p,\ q' \geq q$.
(II). Moreover, after composing with suitable automorphisms of $D^I_{p,q}$ and $D^I_{p',q'}$, f takes the following form:

$$f : z \to \begin{pmatrix} z & 0 \\ 0 & h(z) \end{pmatrix}. \tag{1.1}$$

Here h is a certain holomorphic $(p'-p)\times(q'-q)-$matrix valued function on $D^I_{p,q}$ satisfying that $I_{q'-q} - \overline{h}^t h$ is positive definite on $D^I_{p,q}$.

In the remaining context of the paper, as in [Ch], if a proper map $f : D^I_{p,q} \to D^I_{p',q'}$ satisfies the conclusion of Conjecture 1.8 (i.e., it takes the form (1.1) after composing with automorphisms), then we say f is of diagonal type. It is known that Conjecture 1.8 holds under the additional assumption that f extends smoothly to a neighborhood of a smooth boundary point. This is a consequence of results obtained by Kim-Zaitsev [KZ2] and Kim [K]. (See Corollary 1 in [KZ2] for case (1), and Theorem 1.2 in [K] for case (2)). Moreover, the assumption in (1) or (2) of Conjecture 1.8 cannot be weakened. Indeed, Seo (see page 445 in [S1]) constructed a proper holomorphic map (generalized Whitney map) from $D^I_{r,s}$ to $D^I_{2r-1,2s-1}$, which is not of diagonal type.

By further developing the double fibration ideas introduced in [Ng2], Chan himself [Ch] confirmed part (I) of Conjecture 1.8. He also proved part (II) of Conjecture 1.8 under the condition in (2), while still left open part (II) under the condition in (1). See Theorem 1.3 in [Ch].

In the second part of this paper, we give a complete affirmative answer to Conjecture 1.8 under the condition in (1). Thus our result together with the work of Chan [Ch] leads to the following theorem:

Theorem 1.9. *Conjecture 1.8 holds.*

The paper is organized as follows. Section 2 recaps some notions and results from [HLTX], and proves Theorem 1.1 and Corollary 1.3. In Section 3, we recall some preliminaries from [Ng2, Ch], and give a proof of Theorem 1.9. Corollary 1.3 will play a crucial role in the proof. At the end, we prove a rigidity theorem for proper maps from $D^I_{r,s}$ to $D^I_{s,s}$, which generalizes a result of Tu [T2].

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2 Proofs of Theorem 1.1 and Corollary 1.3

We will prove Theorem 1.1 in Section 2.1, and prove Corollary 1.3 in Section 2.2.

2.1 Proof of Theorem 1.1

The following lemma plays an important role here. For $z = (z_1, \cdots, z_m) \in \mathbb{C}^m$, write $|z|_l^2 = -\sum_{j=1}^l |z_j|^2 + \sum_{j=l+1}^m |z_j|^2$.

Lemma 2.1. *Let l, m, a, b be nonnegative integers such that $m \geq 2, 1 \leq l \leq m-1$. Let $\varphi_1, ..., \varphi_a, \psi_1, ..., \psi_b$ be homogeneous holomorphic polynomials of the same degree in $\mathbb{C}^m$ such that*

$$-\sum_{j=1}^{a} |\varphi_j(z)|^2 + \sum_{j=1}^{b} |\psi_j(z)|^2 = A(z, \bar{z})|z|_l^2, \quad z \in \mathbb{C}^m, \tag{2.1}$$

where $A(z, \bar{z})$ is a real polynomial. Assume one of the following conditions holds:

$$\begin{aligned}
&(1). \quad a < l, a < m-l;\\
&(2). \quad a < l, b < l;\\
&(3). \quad b < m-l, a < m-l;\\
&(4). \quad b < m-l, b < l.
\end{aligned}$$

Then $A(z, \bar{z}) \equiv 0$.

Proof. We can assume $a \geq 1$ and $b \geq 1$. Indeed the conclusion follows easily by checking the zero locus of both sides of (2.1) if $a = 0$ or $b = 0$. We can also assume φ_j's and ψ_j's are not all identically zero, for otherwise the conclusion is trivial.

Denote by $[z] = [z_1, ..., z_m]$ the homogeneous coordinates in $\mathbb{P}^{m-1}$, and define a rational map $F : \mathbb{P}^{m-1} \to \mathbb{P}^{a+b-1}$:

$$F([z]) = [\varphi_1(z), \cdots, \varphi_a(z), \psi_1(z), \cdots, \psi_b(z)].$$

It is a well-defined holomorphic map on $\mathbb{P}^{m-1}$ away from the variety V, where $V \subset \mathbb{P}^n$ denotes the complex analytic variety where φ_j's and ψ_j's all vanish. Recall the generalized sphere is defined as

$$\partial\mathbb{B}_k^{N-1} = \{[w_1, ..., w_N] \in \mathbb{P}^{N-1} : |w_1|^2 + ... + |w_{k+1}|^2 = |w_{k+2}|^2 + ... + |w_N|^2\}.$$

Since $-\sum_{j=1}^a |\varphi_j(z)|^2 + \sum_{j=1}^b |\psi_j(z)|^2 = 0$ when $|z|_l^2 = 0$, we see F gives a holomorphic map that sends an open piece of $\partial\mathbb{B}_{l-1}^{m-1}$ to $\partial\mathbb{B}_{a-1}^{a+b-1}$.

Note the number of the negative and positive eigenvalues of the Levi form of $\partial\mathbb{B}_{a-1}^{a+b-1}$ are $a-1$ and $b-1$. If (1) or (2) holds, then $a < l \leq m-1$ and thus $a-1 < m-2$. If (3) or (4) holds, then $b < m-l \leq m-1$ and thus $b-1 < m-2$. In any case, by Lemma 4.1 in [BH] (or Theorem 1.1 of [BER]), it follows that one of the following two mutually exclusive statements must hold: (I). There exists a neighborhood $V \subseteq \mathbb{P}^{m-1}$ of some open piece of $\partial\mathbb{B}_{l-1}^{m-1}$ such that $F(V) \subseteq \partial\mathbb{B}_{a-1}^{a+b-1}$; (II). F is transversal to $\partial\mathbb{B}_{a-1}^{a+b-1}$ at $F(p)$ for a generic point $p \in \partial\mathbb{B}_{l-1}^{m-1}$.

We claim that Case (II) cannot hold. Indeed, Suppose (II) holds. Then by the existence of such a generically transversal map F, we compare the number of the negative and positive eigenvalues of the Levi forms of $\partial\mathbb{B}_{l-1}^{m-1}$ and $\partial\mathbb{B}_{a-1}^{a+b-1}$ to see one of the following statements must hold:

(a). $l-1 \leq a-1$, and $m-l-1 \leq b-1$;

(b). $l-1 \leq b-1$, and $m-l-1 \leq a-1$.

Indeed , if F preserves the sides of $\partial\mathbb{B}_{l-1}^{m-1}$ and $\partial\mathbb{B}_{a-1}^{a+b-1}$, then (a) holds. If F changes the sides, then (b) holds (see, for instance, [BH]). But neither (a) nor (b) can be true if one of the conditions (1)–(4) holds. This is a contradiction. Thus we must have (I) holds. This implies $A(z, \bar{z}) \equiv 0$, and finishes the proof of Lemma 2.1. ■

We are now at the position to prove Theorem 1.1.

Proof of Theorem 1.1: By Theorem 1.1 in [HLTX], F is an isometric embedding if and only if F is CR transversal at $F(q)$ for a generic point $q \in U \cap \partial\mathbb{B}_l^n$ and F has zero geometric rank near q. The detailed definition of the geometric rank was given in [HLTX] (see Section 3 there) for a CR transversal map from $\partial\mathbb{B}_l^n$ to $\partial\mathbb{B}_{l'}^N$. Roughly speaking, the zero geometric rank at a point $\hat{q} \in F(U \cap \partial\mathbb{B}_l^n)$ is equivalent to the condition that for any $X_{\hat{q}} \in T_{\hat{q}}^{(1,0)} F(\partial\mathbb{B}_l^n)$, the value at $X_{\hat{q}}$ of the CR second fundamental form $\prod(X_{\hat{q}}, X_{\hat{q}}) \in T_{\hat{q}}^{(1,0)}(\partial\mathbb{B}_{l'}^N)/dF(T^{(1,0)}(\partial\mathbb{B}_l^n))$ of $F : \partial\mathbb{B}_l^n \to \partial\mathbb{B}_{l'}^N$ stays in the null cone of the Levi form $\mathcal{L}_{\hat{q}}$ of $\partial\mathbb{B}_{l'}^N$ at $\hat{q}$: $\mathcal{L}_{\hat{q}}(\prod(X_{\hat{q}}, X_{\hat{q}}), \prod(X_{\hat{q}}, X_{\hat{q}})) = 0$. To give a more precise explanation for the notion of geometric rank, it however requires some technical preparation. We will thus postpone it to the end of § 2.1 (see Remark 2.4).

Note by assumption of Theorem 1.1, we have either $l' < n-1$ or $N - l' - 1 < n - 1$. Then by Lemma 4.1 of [BH], F is CR transversal at a generic point $q \in U \cap \partial\mathbb{B}_l^n$. Fix such a point $q = q_0$. It then suffices to show F has zero geometric rank near q_0.

Proposition 2.2. *The map F has zero geometric rank near q_0 along $U \cap \partial\mathbb{B}_l^n$.*

Proof of Proposition 2.2: We first recall some notations from [BH] and [HLTX] which will be needed in the proof. Given $l \geq 1$, We denote by $\delta_{j,l}$ the symbol which takes value -1 when $1 \leq j \leq l$ and 1 otherwise. If $l = 0$, $\delta_{j,0}$ is identically one for all $j \geq 1$. For fixed integers $l' \geq l \geq 1$ and $n \geq 1$, we denote by $\delta_{j,l,l',n}$ the symbol which takes value -1 when $1 \leq j \leq l$ or $n \leq j \leq n + l' - l - 1$ and 1 otherwise. When $l' = l, \delta_{j,l,l,n}$ is the same as $\delta_{j,l}$. Let $m \geq 1$. For two m-tuples $x = (x_1, \cdots, x_m)$, $y = (y_1, \cdots, y_m)$ of complex numbers, we write $\langle x, y\rangle_l = \sum_{j=1}^m \delta_{j,l} x_j y_j$, and $|x|_l^2 = \langle x, \bar{x}\rangle_l$.

Recall for $0 \leq l \leq n-1$, the generalized Siegel upper-half space is defined by

$$\mathbb{S}_l^n = \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \mathrm{Im}(w) > \sum_{j=1}^{n-1} \delta_{j,l} |z_j|^2\}.$$

Its boundary is the standard hyperquadrics: $\mathbb{H}_l^n = \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \mathrm{Im}(w) = \sum_{j=1}^{n-1} \delta_{j,l} |z_j|^2\}$. Similarly for $l \leq l' \leq N-1$, we define

$$\mathbb{S}_{l,l',n}^N = \{(Z, W) \in \mathbb{C}^{N-1} \times \mathbb{C} : \mathrm{Im}(W) > \sum_{j=1}^{N-1} \delta_{j,l,l',n} |Z_j|^2\}.$$

And $\mathbb{S}_{l'}^N, \mathbb{H}_{l'}^N, \mathbb{H}_{l,l',n}^N$ are all defined in a similar manner. Now for $(z, w) = (z_1, \cdots, z_{n-1}, w) \in \mathbb{C}^n$, let $\Psi(z, w) = [i + w, 2z, i - w] \in \mathbb{P}^n$. Then Ψ is the Cayley transformation which biholomorphically maps $\mathbb{S}_l^n$ and its boundary $\mathbb{H}_l^n$ onto $\mathbb{B}_l^n \setminus \{[z_0, \cdots, z_n] : z_0 + z_n = 0\}$ and $\partial\mathbb{B}_l^n \setminus \{[z_0, \cdots, z_n] : z_0 + z_n = 0\}$, respectively.

Composing F with automorphisms of $\mathbb{B}_l^n$ and $\mathbb{B}_{l'}^N$ if necessary, we assume that $q_0 = [1, 0, ..., 0, 1] \in \partial\mathbb{B}_l^n$ and $F(q_0) = [1, 0, ..., 0, 1] \in \partial\mathbb{B}_{l'}^N$. Denote by Ψ the aforementioned Cayley transformation from $\mathbb{S}_l^n$ to $\mathbb{B}_l^n$, and Φ the Cayley transformation from $\mathbb{S}_{l,l',n}^N$ to $\mathbb{B}_{l'}^N$. Then $\widetilde{F} := \Phi^{-1} \circ F \circ \Psi$ is well-defined in a small neighborhood of $0 \in \mathbb{H}_l^n$; and $\tilde{F}$ is side-preserving (i.e., it maps $\mathbb{S}_l^n$ to $\mathbb{S}_{l,l',n}^N$ near 0). Moreover, by the definition of the geometric rank (see Section 3 in [HLTX]), to show F is of geometric rank zero near q_0, it suffices to prove the new map $\widetilde{F}$ has zero geometric rank near 0. To keep notations simple, we will still write the new map as F instead of $\widetilde{F}$. That is, F is now a holomorphic map from a neighborhood V of $0 \in \mathbb{H}_l^n$ to $\mathbb{C}^N$, satisfying

$$F(V \cap \mathbb{S}_l^n) \subseteq \mathbb{S}_{l,l',n}^N \qquad \textit{and} \qquad F(V \cap \mathbb{H}_l^n) \subseteq \mathbb{H}_{l,l',n}^N.$$

By shrinking V if necessary, we can additionally assume $M_1 := V \cap \mathbb{H}_l^n$ is connected and F is CR transversal along M_1.

Next for each $p \in M_1$, we associate it with a map F_p defined as in [BH] and [HLTX]:

$$F_p = \tau_p^F \circ F \circ \sigma_p^0. \tag{2.2}$$

Here $\sigma_p^0 \in \mathrm{Aut}(\mathbb{H}_l^n)$ and $\tau_p^F \in \mathrm{Aut}(\mathbb{H}_{l,l',n}^N)$ are as defined in [BH, HLTX]; see (3.2) in [HLTX] and the paragraph below it. Then F_p is a holomorphic map in a neighborhood of $0 \in \mathbb{C}^n$, which sends an open piece of $\mathbb{H}_l^n$ into $\mathbb{H}_{l,l',n}^N$ with $F_p(0) = 0$. Moreover, $F_p(U \cap \mathbb{S}_l^n) \subseteq \mathbb{S}_{l,l',n}^N$. Let F_p^*, F_p^{**} be the first and second normalizations of F_p, respectively, as in [BH, HLTX]; see (3.9) and (3.13) of [HLTX]. Then as in [HLTX], F_p^{**} map 0 to 0, and maps $\mathbb{H}_l^n$ (respectively, $\mathbb{S}_l^n$) to $\mathbb{H}_{l,l',n}^N$ (respectively, $\mathbb{S}_{l,l',n}^N$) near 0. Write $F_p^{**} = (f_p^{**}, \phi_p^{**}, g_p^{**})$, where f_p^{**} has $n-1$ components, ϕ_p^{**} has $N-n$ components, and g_p^{**} is a scalar function.

We adopt the notations for functions of weighted degree from [Hu1] and [BH]. Parameterize $\mathbb{H}_l^n$ by $(z, \overline{z}, u)$ through the map $(z, \overline{z}, u) \to (z, u + i\sum_{j=1}^{n-1} \delta_{j,l}|z_j|^2)$. We assign the weight of z to be 1, and assign the weight of u (and thus w) to be 2 . For a smooth function $h(z, \bar{z}, u)$ defined in a neighborhood W of 0 in $\mathbb{H}_l^n$, we say it is of quantity $O_{wt}(s)$ for $0 \leq s \in \mathbb{N}$, if $\frac{h(tz, t\bar{z}, t^2 u)}{t^s}$ is bounded for (z, u) on any compact subset of W and t close to 0. Furthermore, for a smooth function $h(z, \bar{z}, u)$ on W, we denote by $h^{(k)}(z, \bar{z}, u)$ the sum of terms of weighted degree k in the Taylor expansion of h at 0. And $h^{(k)}(z, \bar{z}, u)$ also sometimes denotes a weighted homogeneous polynomial of degree k, if h is not specified. When $h^{(k)}(z, \bar{z}, u)$ extends to a holomorphic polynomial of weighted degree k, we write it as $h^{(k)}(z, w)$ or $h^{(k)}(z)$ if it depends only on z.

Under the notations above, by Lemma 2.2 in [BH] (which is also Lemma 3.1 in [HLTX]), F_p^{**} satisfies the following normalization. Here recall $(z, w) = (z_1, \cdots, z_{n-1}, w)$ denotes the coordinates of $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$.

Remark 2.3. *We take the chance to point out that there is a typo in the equation (3.14) in the paper [HLTX]. The right hand side of the equation should be raised to the power* 2 *(the correct version is as in the following equation (2.3)).*

Lemma. *(Lemma 2.2 in [BH]) For each $p \in M_1$, F_p^{**} satisfies the normalization condition:*

$$\begin{cases} f_p^{**} = z + \frac{i}{2} a_p^{**(1)}(z) w + O_{wt}(4) \\ \phi_p^{**} = \phi_p^{**(2)}(z) + O_{wt}(3) \\ g_p^{**} = w + O_{wt}(5), \end{cases}$$

with

$$\langle \bar{z}, a_p^{**(1)}(z) \rangle_l |z|_l^2 = |\phi_p^{**(2)}(z)|_\tau^2, \ \tau = l' - l. \tag{2.3}$$

Remark 2.4. *We briefly recall the notion of geometric rank from [HLTX]. If we write $a_p^{**(1)}(z) = z\mathcal{A}(p)$ for any $(n-1) \times (n-1)$ matrix $\mathcal{A}(p)$, then the geometric rank of F at p is defined as the rank of the matrix $\mathcal{A}(p)$. In particular, F have geometric rank zero at p if and only if $\mathcal{A}(p)$ is the zero matrix. See more details of this definition in Section 3 of [HLTX].*

Set $\mathcal{A}_p(z, \bar{z}) = \langle \bar{z}, a_p^{**(1)}(z) \rangle_l$, which by (2.3) is a real polynomial. Then it follows from (2.3) that

$$\mathcal{A}_p(z, \bar{z}) |z|_l^2 = |\phi_p^{**(2)}(z)|_\tau^2.$$

Write $m = n - 1, a = \tau = l' - l$, and $b = N - n - (l' - l)$. Note if one of the conditions (1)–(4) in Theorem 1.1 holds, then one of the conditions (1)–(4) holds in Lemma 2.1. Then by Lemma 2.1, we see $\mathcal{A}_p(z, \bar{z}) \equiv 0$, and thus F has geometric rank zero at p. Since p is an arbitrary point close to 0, we conclude that F has zero geometric rank near 0 along M_1. This proves Proposition 2.2. ∎

This finishes the proof of Theorem 1.1. ∎

2.2 Proof of Corollary 1.3

The following lemma will imply Corollary 1.3.

Lemma 2.5. *Let F be a rational proper map from $\mathbb{P}^n$ to $\mathbb{P}^N$ with I its set of indeterminacy and $N \geq n$. If F is an isometric map from $(\mathbb{B}_l^n \setminus I, \omega_{\mathbb{B}_l^n})$ to $(\mathbb{B}_{l'}^N, \omega_{\mathbb{B}_{l'}^N})$ with $l' \geq l \geq 1$, then F is a linear embedding from $\mathbb{P}^n$ to $\mathbb{P}^N$. Moreover, there exists $h \in Aut(\mathbb{B}_{l'}^N)$ such that*

$$h \circ F([z]) = [z_0, ..., z_l, 0, ..., 0, z_{l+1}, ..., z_n, 0, ..., 0],$$

for $[z] = [z_0, ..., z_l, z_{l+1}, ..., z_n] \in \mathbb{P}^n$, where the first zero tuple has $l' - l$ components.

Proof. By Theorem 2.1 in [HLTX], we conclude, by composing automorphisms of $\mathbb{B}^n_l$ and $\mathbb{B}^N_{l'}$, F equals to the following map:

$$[z] = [z_1, ..., z_n] \to [z_1, ..., z_l, \phi, z_{l+1}, ..., z_n, \psi],$$

where ϕ has $l' - l$ components ψ has $N - n - (l' - l)$ components and satisfy $\|\phi\| = \|\varphi\|$ at points where they are defined. Moreover, by the rationality assumption, ϕ, ψ are rational maps in $z = (z_1, \cdots, z_n)$.

Thus we can write $\phi = \frac{p_1}{q}, \psi = \frac{p_2}{q}$. Here p_1, p_2 are polynomial maps in z, such that their nonzero components are all homogeneous polynomials with the same degree. And $q \neq 0$ is a homogeneous polynomial maps in z. They satisfy the following conditions:

(A). $\|p_1(z)\| = \|p_2(z)\|, \quad \forall z \in \mathbb{C}^n$;

(B). p_1, p_2 and q have only trivial common factors ($\Leftrightarrow p_1$ and q are coprime, by (A));

(C). $\deg q = \deg p_1 - 1 = \deg p_2 - 1$ if p_1 and p_2 are not identically zero.

Note we can rewrite

$$F([z]) = [z_1 q, ..., z_l q, p_1, z_{l+1} q, ..., z_n q, p_2]. \tag{2.4}$$

The set of indeterminacy I of F is given by

$$I = \{[z] \in \mathbb{P}^n : p_1(z) = 0, p_2(z) = 0, q(z) = 0\} = \{[z] \in \mathbb{P}^n : p_1(z) = 0, q(z) = 0\}.$$

Note I is of codimension at least 2 in $\mathbb{P}^n$ (which is indeed known as a general fact without using the above explicit formula). We claim q is a constant function. Otherwise, recall $l \geq 1$, we can find a point $[z^*] = [z^*_0, z^*_1, 0, ..., 0] \in \mathbb{B}^n_l$ such that $q(z^*) = 0$. Since I is of codimension at least 2, we can find a point $[\tilde{z}] \in \mathbb{B}^n_l$ close to $[z^*]$ such that $q(\tilde{z}) = 0$, and $[\tilde{z}] \notin I$. By the equation (2.4), $F([\tilde{z}]) \in \partial\mathbb{B}^N_{l'}$. This contradicts with the definition of rational proper maps from $\mathbb{B}^n_l$ to $\mathbb{B}^N_{l'}$. Hence we must have q is a constant function. Consequently, we have either $\deg p_1 = \deg p_2 = 1$, or p_1 and p_2 are identically zero. Thus F is a linear embedding from $\mathbb{P}^n$ to $\mathbb{P}^N$. To prove the last conclusion of the lemma, by the linearity of f, we write $F([z]) = [z]A$ for some $A \in M(n+1, N+1; \mathbb{C})$. Since F is a proper map from $\mathbb{B}^n_l$ to $\mathbb{B}^N_{l'}$, we see that $AI_{l'+1,N+1}\overline{A}^t = \lambda I_{l+1,n+1}$ for some $\lambda \in \mathbb{R}$. Since F maps $\mathbb{B}^n_l$ to $\mathbb{B}^N_{l'}$, we have $\lambda > 0$. By scaling A if necessary, we can assume $\lambda = 1$. Then it follows that there exists some matrix $U \in M(N+1, N+1; \mathbb{C})$ such that $UI_{l'+1,N+1}\overline{U}^t = I_{l'+1,N+1}$ and AU takes the following forms (see, for instance, page 386 in [BH]):

$$AU = (C \quad D), \qquad \text{with } C = \begin{pmatrix} I_{l+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_{(n+1)\times(l'+1)} \quad \text{and} \qquad D = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ I_{n-l} & \mathbf{0} \end{pmatrix}_{(n+1)\times(N-l')}.$$

Here the **0** symbols denote the zero matrices of appropriate (and possibly different) sizes. This proves the last assertion of the lemma. ∎

Finally we prove Corollary 1.3.

Proof of Corollary 1.3: Let $U = \mathbb{P}^n \setminus I$, where I is the set of indeterminacy of F. Then we have $U \cap \mathbb{B}^n_l$ is connected and $F(U \cap \mathbb{B}^n_l) \subseteq \mathbb{B}^N_{l'}$, $F(U \cap \partial\mathbb{B}^n_l) \subseteq \partial\mathbb{B}^N_{l'}$. Then by Theorem 1.1, F is an isometric map from $(U \cap \mathbb{B}^n_l, \omega_{\mathbb{B}^n_l})$ to $(\mathbb{B}^N_{l'}, \omega_{\mathbb{B}^N_{l'}})$. Then the conclusion follows from Lemma 2.5. ∎

3 Proof of Theorem 1.9

In this section, to make the notations easier, we will use a different notation $D_{r,s}$ to denote the generalized unit ball. More precisely, we write

$$D_{r,s} = \mathbb{B}^{r+s-1}_{r-1} = \{[z_1, ..., z_{r+s}] \in \mathbb{P}^{r+s-1} : \sum_{j=1}^{r} |z_j|^2 > \sum_{j=r+1}^{r+s} |z_j|^2\}.$$

Furthermore, in the following, we will use the notation of $[A, B]_r \in D_{r,s}$ to indicate that A is a $r-$dimensional row vector, and B is an $s-$dimensional row vector. Note they satisfy $A\overline{A}^t > B\overline{B}^t$. Before we proceed to prove Theorem 1.9, we first recall some preliminaries about holomorphic double fibrations from [Ng2]. Consider the following double fibration:

$$D_{r,s} \xleftarrow{\pi^1_{r,s}} \mathbb{P}^{r-1} \times D^I_{r,s} \xrightarrow{\pi^2_{r,s}} D^I_{r,s}$$

Here

$$\pi^1_{r,s}([X], Z) = [X, XZ]_r, \quad \text{for } [X] \in \mathbb{P}^{r-1}, Z \in D^I_{r,s}.$$

And $\pi^2_{r,s}$ is the standard projection onto $D_{r,s}$. For $x = [A, B]_r \in D_{r,s} \subseteq \mathbb{P}^{r+s-1}$ and $Z \in D^I_{r,s}$, their fibral images are defined, respectively, as the following:

$$x^\sharp = [A, B]^\sharp_r = \pi^2_{r,s}\big((\pi^1_{r,s})^{-1}([A, B]_r)\big) \subseteq D^I_{r,s}; \qquad Z^\sharp = \pi^1_{r,s}\big((\pi^2_{r,s})^{-1}(Z)\big) \subseteq D_{r,s}.$$

As shown in [Ng2], we indeed have the following formulas for the fibral images:

$$x^\sharp = \{Z \in D^I_{r,s} : AZ = B\}; \qquad Z^\sharp = \{[A, AZ]_r \in D_{r,s} : [A] \in \mathbb{P}^{r-1}\}.$$

Let $f : D^I_{q,p} \to D^I_{q',p'}$ be a holomorphic map. We say f is fibral-image-preserving with respect to the double fibrations:

$$
\begin{aligned}
D_{q,p} \xleftarrow{\pi^1_{q,p}} \mathbb{P}^{q-1} \times D^I_{q,p} \xrightarrow{\pi^2_{q,p}} D^I_{q,p}, \\
D_{q',p'} \xleftarrow{\pi^1_{q',p'}} \mathbb{P}^{q'-1} \times D^I_{q',p'} \xrightarrow{\pi^2_{q',p'}} D^I_{q',p'}.
\end{aligned}
\tag{3.1}
$$

if for any $[A,B]_q \in D_{q,p}$, we have $f([A,B]^\sharp_q) \subseteq [C,D]^\sharp_{q'}$ for some $[C,D]_{q'} \in D_{q',p'}$. Furthermore, let U be an open subset of $D_{q,p}$. We say a holomorphic map $g : U \to D_{q',p'}$ is a moduli map of f on U if $f([A,B]^\sharp_q) \subseteq g([A,B]_q)^\sharp$ for all $[A,B]_q \in U$.

The following proposition, which is based on Proposition 7.2 in [Ch], will be crucial for the proof of Theorem 1.9.

Proposition 3.1. *Let $f : D^I_{q,p} \to D^I_{q',p'}$ be a proper holomorphic map where $p \geq q \geq 2$, and $3 \leq q' < p$. Then the following statements hold.*

(a). Then f is fibral-image-preserving with respect to the double fibrations (3.1). And there exists a holomorphic map $g : U \subseteq D_{q,p} \to D_{q',p'}$ such that g is a moduli map of f on U, where U is a dense open subset of $D_{q,p}$. Furthermore, g extends to a rational map from $\mathbb{P}^{p+q-1}$ to $\mathbb{P}^{p'+q'-1}$. Write I for the set of indeterminacy of g, we have

$$g(\partial D_{q,p} \setminus I) \subseteq \partial D_{q',p'}.$$

And we have $p' \geq p, q' \geq q$.

(b). We have g maps $D_{q,p} \setminus I$ to $D_{q',p'}$. Consequently, g is a rational proper map from $D_{q,p}$ to $D_{q',p'}$.

(c). If either $p' < 2p-1$ or $q' < 2q-1$, then f is of diagonal type.

Proof. Note part (a) was already established in [Ch]. We remark that one can also see $p' \geq p, q' \geq q$ from the property of g. Indeed, g maps some open piece of $\partial D_{q,p}$ to $\partial D_{q',p'}$. Since $p > q'$, the number of negative eigenvalues of the Levi form of $\partial D_{q',p'}$, $q'-1$, is less than $p+q-2$. Thus by Lemma 4.1 in [BH] (or Theorem 1.1 in [BER]), g is CR transversal at $g(z)$ for a generic point z of $\partial D_{q,p}$. Furthermore, g preserves the sides of the hypersurfaces, i.e., g maps $U \cap D_{q,p}$ to $D_{q',p'}$ for some small neighborhood U of z. Thus, by comparing the number of the positive and negative eigenvalues of the Levi forms of $\partial D_{q,p}$ and $\partial D_{q',p'}$, we see $p' \geq p$ and $q' \geq q$.

To prove part (b), as in [S1] and [Ch], we extend the definition of double fibrations to topological closures of the type I classical domains in complex Euclidean spaces and topological closures of the generalized complex balls in complex projective spaces. That is, consider the following double fibration:

$$\overline{D_{r,s}} \xleftarrow{\tilde{\pi}^1_{r,s}} \mathbb{P}^{r-1} \times \overline{D^I_{r,s}} \xrightarrow{\tilde{\pi}^2_{r,s}} \overline{D^I_{r,s}} \tag{3.2}$$

Here $\tilde{\pi}^1_{r,s}$ and $\tilde{\pi}^2_{r,s}$ are defined similarly as $\pi^1_{r,s}$ and $\pi^2_{r,s}$:

$$\tilde{\pi}^1_{r,s}([X], Z) = [X, XZ]_r, \qquad [X] \in \mathbb{P}^{r-1}, Z \in \overline{D^I_{r,s}},$$

and $\tilde{\pi}^2_{r,s}$ is the standard projection onto $\overline{D_{r,s}}$. Under the double fibration (3.2), for $x = [A, B]_r \in \overline{D_{r,s}} \subseteq \mathbb{P}^{r+s-1}$ and $Z \in \overline{D^I_{r,s}}$, their fibral images are again defined by

$$x^* = [A, B]^*_r = \tilde{\pi}^2_{r,s}\big((\tilde{\pi}^1_{r,s})^{-1}([A, B]_r)\big) \subseteq \overline{D^I_{r,s}}; \quad Z^* = \tilde{\pi}^1_{r,s}\big((\tilde{\pi}^2_{r,s})^{-1}(Z)\big) \subseteq \overline{D_{r,s}}$$

Similarly as before, we have the following formulas:

$$x^* = \{Z \in \overline{D^I_{r,s}} : AZ = B\}; \qquad Z^* = \{[A, AZ]_r \in \overline{D_{r,s}} : [A] \in \mathbb{P}^{r-1}\}.$$

We see from the above formulas that if $x \in \partial D_{r,s}$, then we have $x^* \subseteq \partial D^I_{r,s}$; while for $Z \in \partial D^I_{r,s}$, Z^* might not be contained in $\partial D_{r,s}$. Note by part (a), with respect to (3.1) and the following double fibrations:

$$\overline{D_{q,p}} \xleftarrow{\tilde{\pi}^1_{q,p}} \mathbb{P}^{q-1} \times \overline{D^I_{q,p}} \xrightarrow{\tilde{\pi}^2_{q,p}} \overline{D^I_{q,p}},$$
$$\overline{D_{q',p'}} \xleftarrow{\tilde{\pi}^1_{q',p'}} \mathbb{P}^{q'-1} \times \overline{D^I_{q',p'}} \xrightarrow{\tilde{\pi}^2_{q',p'}} \overline{D^I_{q',p'}};$$

we have for all $x \in U$,

$$f(x^\sharp) = f(x^* \cap D^I_{q,p}) \subseteq g(x)^\sharp = g(x)^* \cap D^I_{q',p'}.$$

Fix $y \in D_{q,p} - I$. Since U is a dense open subset of $D_{q,p}$, there exists a sequence $\{y_j\}_{j=1}^\infty \subseteq U$ such that $y_j \to y$. Note we can choose $Z_j \in y_j^\sharp \subseteq D^I_{q,p}$ such that $Z_j \to Z$ for some $Z \in y^\sharp \subseteq D^I_{q,p}$. Note we have $f(Z_j) \to f(Z)$. On the other hand, since $g(y_j) \to g(y)$, the limit set of any sequence $\{W_j\}_{j=1}^\infty \subseteq D^I_{q',p'}$ with $W_j \in (g(y_j))^\sharp$ must be contained in $(g(y))^* \subseteq \overline{D^I_{q',p'}}$.

By part (a), $f(Z_j) \in f(y_j^\sharp) \subseteq (g(y_j))^\sharp$. Thus the limit $f(Z)$ of the sequence $f(Z_j)$ must lie in $(g(y))^*$. Suppose $g(y) \in \partial D_{q',p'}$. Then as discussed above, $(g(y))^* \in \partial D^I_{q',p'}$. It will be a contradiction, as f maps $Z \in D^I_{q,p}$ to $D^I_{q',p'}$. This proves part (b).

Note the case $q' < 2q - 1$ of part (c) was already proved in Chan [Ch]. We will anyway give a unified proof for the two cases of part (c) here. To do this, we use the conclusion in part (b) that g is a rational proper map from $D_{q,p}$ to $D_{q',p'}$ in the sense of Definition 1.2. Recall $D_{q,p} = \mathbb{B}^{p+q-1}_{q-1}$. Write $l := q-1, n := p+q-1$, and we have $n \geq 3, 1 \leq l \leq n-2$. Similarly, recall $D_{q',p'} = \mathbb{B}^{p'+q'-1}_{q'-1}$. Write $l' = q'-1, N = p'+q'-1$. By part (a), we have $l' \geq l, N \geq n$.

If $p' < 2p-1$, then $N - l' = p' < 2n - 2l - 1 = 2p - 1$ and $l' = q' - 1 < p - 1 < n - 1$. Therefore the condition (3) in Corollary 1.3 (i.e., in Theorem 1.1) holds. Similarly, if $q' < 2q-1$, then $l' = q'-1 < 2q-2 = 2l$ and $l' = q'-1 < p-1 < n-1$. Therefore the condition (1) in Corollary 1.3 (i.e., in Theorem 1.1) holds.

Hence by Corollary 1.3, in either case of part (c), g is indeed a linear embedding from $\mathbb{P}^{p+q-1}$ to $\mathbb{P}^{p'+q'-1}$. Then it follows from the argument in [Ch] (see Lemma 6.1 and Lemma 6.5 in [Ch]) that f is of diagonal type. This proves part (c) and finishes the proof of Proposition 3.1. ∎

We are ready to prove Theorem 1.9.

Proof of Theorem 1.9: Once we have Proposition 3.1, the proof of Theorem 1.9 is similar to that of Theorem 1.3 in [Ch]. We sketch the proof here. Note the rank of $D^I_{q,p}$ equals $q \geq 2$. Suppose $q' \leq 2$. Then the rank of $D^I_{q',p'}$ is at most 2. Then by Tsai [Ts], F is a standard map (see [Ts], in particular, F is of diagonal type.) Moreover, $q' = q = 2, p' \geq p$. It remains to consider the case $q' \geq 3$. First by Proposition 3.1 part (a), we have $p' \geq p, q' \geq q$. Then similarly as in Chan [Ch], we define $F^\dagger : D^I_{q,p} \to D^I_{q',p'}$ by $F^\dagger(W) = (F(W^t))^t$ for $W \in D^I_{q,p}$. Here t denotes the transpose of a matrix. Then $f = F^\dagger$ is a proper holomorphic map from $D^I_{q,p}$ to $D^I_{q',p'}$. By Proposition 3.1, $F^\dagger$ is of diagonal type. Finally, by Lemma 4.2 in [Ch], F is of diagonal type as well. This finishes the proof of Theorem 1.9. ∎

To conclude this section, we prove the following consequence of [Ng2, Ch]. The result generalizes a theorem of Tu [T2]. Note when $r = s - 1$, Proposition 3.2 is reduced to Theorem 1.1 in [T2]. It also generalizes Theorem 1.3 of [Ng2] in the case $r' = s$.

Proposition 3.2. *Let $s > r \geq 2$. Then every proper holomorphic map $f : D^I_{r,s} \to D^I_{s,s}$ is standard. That is, f is a totally geodesic isometric embedding (up to normalization constants) with respect to the Bergman metrics.*

The proposition has an immediate consequence on the non-existence of holomorphic proper maps.

Corollary 3.3. *There exist no proper holomorphic mappings from $D^I_{r,s+1}$ to $D^I_{s,s}$ for $s \geq r \geq 2$.*

Remark 3.4. *Note Corollary 3.3 fails if $r = 1$ and $s \geq 2$. Indeed there is a proper holomorphic map from $D^I_{1,s+1} = \mathbb{B}^{s+1}$ to $D^I_{s,s}$ (see [T2]).*

Proof of Proposition 3.2: As in Ng [Ng2], we let $f^t : D^I_{r,s} \to D^I_{s,s}$ be the induced map defined by $f^t(Z) = (f(Z))^t$. It follows from Corollary 5.6 of [Ng2] that either f or f^t is fibral-image-preserving. Thus, replacing f by f^t if necessary, we can assume f is fibral-image-preserving. Then it follows from Theorem 6.11 and Proposition 6.12 in [Ch] that there are some 'global' moduli maps of f (the existence of the local moduli map follows from Proposition 2.15 in [Ng2]) with respect to the double fibrations:

$$D_{r,s} \xleftarrow{\pi^1_{r,s}} \mathbb{P}^{r-1} \times D^I_{r,s} \xrightarrow{\pi^2_{r,s}} D^I_{r,s},$$

$$D_{s,s} \xleftarrow{\pi^1_{s,s}} \mathbb{P}^{s-1} \times D^I_{s,s} \xrightarrow{\pi^2_{s,s}} D^I_{s,s}.$$

More precisely, there exist a set of linearly independent holomorphic maps $g_j : U \subseteq D_{r,s} \to D_{s,s}, 1 \leq j \leq k_0 + 1$, where $U \subseteq D_{r,s}$ is the complement of some complex subvarieties of $D_{r,s}$, such that every g_j is a moduli map of f on U.

Indeed, as in [Ch], write for $0 \leq i \leq s - 1$,

$$\mathcal{U}_i = \{x \in D_{r,s} : f(x^\sharp) \text{ lies in a } (s-i-1, s) - \text{subspace of } D^I_{r,s}\}.$$

By the argument of Chan (page 33-36, [Ch]), there exists some $0 \leq k_0 \leq r - 2$ such that $D_{r,s} = \mathcal{U}_{k_0}$ and $\mathcal{U}_{k_0+1} \subseteq D_{r,s}$ is a proper complex analytic subvariety. Set $U = D_{r,s} \setminus \mathcal{U}_{k_0+1}$. This implies for every $x \in U$, $f(x^\sharp)$ lies in a unique $(s - k_0 - 1, s)-$subspace of $D^I_{s,s}$. Denote by $G(p, q)$ the complex Grassmannian of $p-$dimensional complex linear subspaces of $\mathbb{C}^{p+q}$, and by $M(p, q; \mathbb{C})$ the space of $p \times q$ matrices with complex entries. Define

$$D^l_{s,s} = \big\{[W', W''] \in G(l, 2s - l) : W'\overline{W'}^t > W''\overline{W''}^t\big\}.$$

Here in the above, $W', W'' \in M(l, s; \mathbb{C})$. By Proposition 6.12 in [Ch], there exists a meromorphic map $g = [g^t_1, \cdots, g^t_{k_0+1}]^t$ from $D_{r,s}$ to $D^l_{s,s}$, with $l = k_0 + 1$, such that $f(x^\sharp) \subseteq (g_j(x))^\sharp$ for all $x \in U, 1 \leq j \leq k_0 + 1$.

Then by Hartog's extension, each g_j extends to a rational map from $\mathbb{P}^{r+s-1}$ to $\mathbb{P}^{2s-1}$ (see, for example, [Ng1, Ng2]).

The next claim follows from the arguments on page 320-321 of [Ng2] and in the proof of Proposition 6.2 in [Ch]. Although the settings are not precisely the same, their arguments still apply here. For the convenience of the readers, we sketch the proof here anyway.

Claim 3.5. *There exists some $1 \leq j_0 \leq k_0+1$ such that g_{j_0} maps $\partial D_{r,s} \setminus I_{j_0}$ to $\partial D_{s,s}$, where I_j is the set of indeterminacy of $g_j, 1 \leq j \leq k_0 + 1$.*

Proof of Claim 3.5: Suppose none of $g_j, 1 \leq j \leq k_0+1$, maps $\partial D_{r,s} \setminus I_j$ to $\partial D_{s,s}$. Then every g_j maps a generic boundary point $p \in \partial D_{r,s}$ to $D_{s,s}$. In what follows, as in the proof of Theorem 1.3 in [Ng2], we will make use of the following double fibration:

$$\mathbb{P}^{r+s-1} \xleftarrow{\hat{\pi}^1_{r,s}} \mathcal{F}^{1,r}_{r+s} \xrightarrow{\hat{\pi}^2_{r,s}} G(r,s).$$

Here $\mathcal{F}^{1,r}_{r+s} = \{(J,K) \in \mathbb{P}^{r+s-1} \times G(r,s) : J \subseteq K\}$. And $\hat{\pi}^1_{r,s}, \hat{\pi}^2_{r,s}$ are the projections to the first and second factor, respectively. For any $x \in G(r,s)$, we write $x^\sharp := \hat{\pi}^1_{r,s}((\hat{\pi}^2_{r,s})^{-1}(x))$. For $y \in \mathbb{P}^{r+s-1}$, write $y^\sharp := \hat{\pi}^2_{r,s}((\hat{\pi}^1_{r,s})^{-1}(y))$. As in [Ng2], $y^\sharp$ is called a $(r-1,s)-$subspace of $G(r,s)$. More generally, one can define the $(k,l)-$subspace of $G(r,s)$ for $1 \leq k \leq r$ and $1 \leq l \leq s$ in a similar manner (see [Ng2]).

As in the proof of Theorem 1.3 of [Ng2], by using Fatou's theorem and taking radical limit, for almost every choice of $p \in \partial D_{r,s}$, f can be extended to $p^\sharp \cap \partial D^I_{r,s}$. Fix such a point $p \in \partial D_{r,s}$. Then as on page 320 in [Ng2], we choose a special one-parameter family of $(r-1,s-1)-$subspaces $\lambda(t) \subseteq G(r,s)$ of a special form and a curve $\Lambda(t) \subseteq \overline{D_{r,s}}$ where $t \in \mathbb{C}$ and $|t| \leq 1$, with the property that $\lambda(t) \subseteq (\Lambda(t))^\sharp$ for every t, $\Lambda(1) = p$, and $\lambda(1) \cap \partial D^I_{r,s} = p^\sharp \cap \partial D^I_{r,s} = (\Lambda(1))^\sharp \cap \partial D^I_{r,s}$. By taking limit as $t \to 1$ as on page 320 in [Ng2], we get

$$f(p^\sharp \cap \partial D^I_{r,s}) \subseteq \bigcap_{j=1}^{k_0+1} (g_j(p))^\sharp \cap \partial D^I_{s,s}. \tag{3.3}$$

Recall $g(z) = [g_1^t(z), \cdots, g^t_{k_0+1}(z)]^t \in D^l_{s,s}$ for all $z \in U$. Then by making p a generic point on $\partial D_{r,s}$, we have $\bigcap_{j=1}^{k_0+1} (g_j(p))^\sharp \cap D^I_{s,s}$ is equivalent, under the action of $SU(s,s)$, to the subspace

$$\left\{ \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ z_{k_0+2,1} & \cdots & z_{k_0+2,s} \\ \vdots & \ddots & \vdots \\ z_{s,1} & \cdots & z_{s,s} \end{pmatrix} \in D^I_{s,s} \right\},$$

where there are (k_0+1) rows of zeros. On the other hand, the closure of every maximal holomorphic boundary component of the above subspace is, up to automorphism, the following subspace:

$$\left\{ \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & z_{k_0+3,2} & \cdots & z_{k_0+3,s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & z_{s,2} & \cdots & z_{s,s} \end{pmatrix} \in \partial D^I_{s,s} \right\}. \tag{3.4}$$

Again there are (k_0+1) rows of zeros in the above. As in [Ng2], note for any holomorphic map from a complex manifold to a Euclidean space such that the image lies inside the boundary of a bounded symmetric domain Ω, the image must lie in the closure of a maximal holomorphic boundary component of Ω. Hence by (3.3) and (3.4), f maps the boundary $(r-1,s-1)-$subspace $p^\sharp \cap \partial D^I_{r,s}$ into $Y \cap \partial D^I_{s,s}$ for some $(s-k_0-2, s-1)-$subspace Y of $G(s,s)$. By a standard maximum principle argument (see [MT]), we see f maps the $(r-1,s-1)-$subspace $\lambda(t) \cap \partial D^I_{r,s}$ into some $(s-k_0-2,s-1)-$subspace of $D^I_{s,s}$ for every $|t|<1, t\in\mathbb{C}$. Note every $(r-1,s-1)-$subspace of $D^I_{r,s}$ is equivalent to some $\lambda(t)\cap\partial D^I_{r,s}$ under the action of $SU(r,s)$. And the boundary point p we fixed at the beginning can be a generic point on $\partial D_{r,s}$. It follows that a generic $(r-1,s-1)-$subspace of $D^I_{r,s}$, and by continuity, every $(r-1,s-1)-$subspace of $D^I_{r,s}$ is mapped to a $(s-k_0-2,s-1)-$subspace of $D^I_{s,s}$(The readers are referred to [Ng2] for more details of the above arguments). This is, however, a contradiction with the following lemma.

Lemma 3.6. *Let r,s and $f: D^I_{r,s} \to D^I_{s,s}$ be as in Proposition 3.2 and k_0 be as above. Let $X_{r-1,s}$ be a $(r-1,s)-$subspace of $D^I_{r,s}$ such that $f(X_{r-1,s})$ lies in a unique $(s-k_0-1,s)-$subspace $Y_{s-k_0-1,s}\subseteq D^I_{s,s}$. Then f cannot map every $(r-1,s-1)-$subspace contained in $X_{r-1,s}$ into an $(s-k_0-2,s)-$subspace.*

Proof of Lemma 3.6: The proof is essentially the same as that of Lemma 6.3 in [Ch]. We omit the proof. ∎

Hence we must have $g := g_{j_0}$ maps $\partial D_{r,s}\setminus I_{j_0}$ to $\partial D_{s,s}$ for some $1\le j_0\le k_0+1$. This finishes the proof of Claim 3.5. ∎

We continue to prove Proposition 3.2. Once we know g is a rational map from $U\subseteq D_{r,s}$ to $D_{s,s}$ and sends $\partial D_{r,s}\setminus I$ to $\partial D_{s,s}$. It follows from Baouendi-Huang [BH] (see Theorem 1.4 in [BH] or Theorem 4.5 in [Ng2]. Note this result indeed also follows from the case (4) of our Theorem 1.1) that g extends to a linear embedding of $\mathbb{P}^{r+s-1}$ into $\mathbb{P}^{2s-1}$. Then by the same argument as in [Ng2] (see page 322 in [Ng2]), f is standard. This proves Proposition 3.2. ∎

The proof of Corollary 3.3 is similar to that of Corollary 1.2 in [T2].

Proof of Corollary 3.3: Suppose there exists a proper holomorphic mapping f from $D^I_{r,s+1}$ to $D^I_{s,s}$. Fix any $(r,s)-$subspace $X_{r,s} \approx D^I_{r,s}$ of $D^I_{r,s+1}$. Then $f|_{X_{r,s}} : D^I_{r,s} \to D^I_{s,s}$ is a proper and by Proposition 3.2, $f|_{X_{r,s}}$ is standard. This implies $f : D^I_{r,s+1} \to D^I_{s,s}$ maps every minimal disc (see [T2] for the definition) to a minimal disc. By [Ts] (see Proposition 2.2 in [T2]), f is standard. In particular, f induces a standard embedding from $\mathbb{B}^{s+1} \subseteq D^I_{r,s+1}$ to $D^I_{s,s}$. This is impossible and thus gives a contradiction. ■

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