Chern-Moser-Weyl Tensor and Embeddings into Hyperquadrics

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Dedicated to our friend Dick Wheeden

1 Introduction

A central problem in Mathematics is the classification problem. Given a set of objects and an equivalence relation, loosely speaking, the problem asks how to find an accessible way to tell whether two objects are in the same equivalence class. A general approach to this problem is to find a complete set of (geometric, analytic or algebraic) invariants. In the subject of Several Complex Variables and Complex Geometry, a fundamental problem is to classify complex manifolds or more generally, normal complex spaces under the action of biholomorphic transformations. When the normal complex spaces are open and have strongly pseudo-convex boundary, by the Fefferman-Bochner theorem, one needs only to classify the corresponding boundary strongly pseudoconvex CR manifolds under the application of CR diffeomorphisms. The celebrated Chern-Moser theory is a theory which gives two different constructions of a complete set of invariants for such a classification problem. Among various aspects of the Chern-Moser theory (especially the geometric aspect of the theory), the Chern-Moser-Weyl tensor plays a key role. However, this trace-free tensor is defined in a very complicated manner. This makes it hard to apply in the applications. The majority of first several sections in this article surveys some work done in papers of Chern-Moser [CM], Huang-Zhang [HZh], Huang-Zaitsev [HZa]. Here, we give a simple and more accessible account on the Chern-Moser-Weyl tensor. We also make an immediate application of the monotonicity property for this tensor to the study of CR embedding problem for the positive signature case.

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In the last section of this paper, we present new materials. We will show that the family of compact strongly pseudo-convex algebraic hypersurfaces constructed in [HLX] cannot be locally holomorohically embedded into a sphere of any dimension. The argument is based on the rationality result established in [HLX] and the Segre geometry associated with such a family. This gives a negative answer to a long standing folklore conjecture concerning the embeddability of compact strongly pseudo-convex algebraic hypersurfaces into a sphere of sufficiently high dimension. For an extensive discussion on the history on the CR embeddability into spheres, we refer the reader to the introduction section of a recent joint paper of the first author with Zaistev [HZa].

2 Chern-Moser-Weyl tensor for a Levi non-degenerate hypersurface

In this article, we assume that the CR manifolds under consideration are already embedded as hypersurfaces in the complex Euclidean spaces. We first consider the case where the manifolds are even Levi non-degenerate.

We use $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ for the coordinates of $\mathbb{C}^{n+1}$. We always assume that $n \geq 2$, for otherwise the Chern-Moser-Weyl tensor is identically zero. In that setting, one has to consider the Cartan curvature functions instead, which we will not touch in this article.

Let M be a smooth real hypersurface. We say that M is Levi non-degenerate at $p \in M$ with signature $\ell \leq n/2$ if there is a local holomorphic change of coordinates, that maps p to the origin, such that in the new coordinates, M is defined near 0 by an equation of the form:

$$r = v - |z|^2_\ell + o(|z|^2 + |zu|) = 0 \tag{1}$$

Here, we write $u = \Re w, v = \Im w$ and $< a, \bar{b} >_\ell = -\sum_{j\leq \ell} a_j \bar{b}_j + \sum_{j=\ell+1}^{n} a_j \bar{b}_j, |z|^2_\ell = < z, \bar{z} >_\ell$. When $\ell = 0$, we regard $\sum_{j\leq \ell} a_j = 0$.

Assume that M is Levi non-degenerate with the same signature ℓ at any point in M. For a point $p \in M$, a real non-vanishing 1-form θ_p at $p \in M$ is said to be appropriate contact form at p if θ_p annihilates $T^{(1,0)}_p + T^{(0,1)}_p M$ and the Levi form L_{θ_p} associated with θ_p at $p \in M$ has ℓ negative eigenvalues and $n - \ell$ positive eigenvalues. Here we recall the definition of the Levi-form L_{θ_p} at p as follows: We first extend θ_p to a smooth 1-form θ near p such that $\theta|_q$ annihilates $T^{(1,0)}_q + T^{(0,1)}_q M$ at any point $q \approx p$. For any $X_\alpha, X_\beta \in T^{(1,0)}_p$, we define

$$L_{\theta_p}(X_\alpha, X_\beta) := -i < d\theta|_p, X_\alpha \wedge \overline{X}_\beta > . \tag{2}$$

One can easily verify that L_{θ_p} is a well-defined Hermitian form in the tangent space of type $(1,0)$ of M at p, which is independent of the choice of the extension of the 1-form θ. In the literature, any smooth non-vanishing 1-form θ along M is called a smooth contact form, if $\theta|_q$ annihilates $T^{(1,0)}_q M$ for any $q \in M$. If $\theta|_q$ is appropriate at $q \in M$, we call θ an appropriate smooth contact 1-form along M. Write E_p for the set of appropriate contact 1-forms at p defined above, and E for the disjoint union of E_p. Then two elements in E_p are proportional by a positive constant for the case of $\ell < n/2$; and are proportional by a non zero constant when $\ell = n/2$. There is a natural smooth structure over E which makes E into a R^+ fiber bundle over M when $\ell < n/2$, or a R^*-bundle over M when $\ell = n/2$. When M is defined near 0 by an equation of the form as in (1), then $i\partial r$ is an appropriate contact form of M near 0. In particular, for any appropriate contact 1-form θ_0 at $0 \in M$, there is a constant $c \neq 0$ such that $\theta_0 = ic\partial r|_0$. And $c > 0$ when $\ell < n/2$. Applying further a holomorphic change of coordinates $(z,w) \to (\sqrt{|c|}z, cw)$ and the permutation transformation $(z_1, \cdots, z_n, w) \to (z_n, \cdots, z_1, w)$ if necessary, we can simply have $\theta_0 = i\partial r|_0$. Assign the weight of $z, \overline{z}$ to be 1 and that of u, v, w to be 2. We say $h(z,\overline{z},u) = o_{wt}(k)$ if $\frac{h(tz,\overline{tz},t^2u)}{t^k} \to 0$ uniformly on compact sets in (z,u) near the origin. We write $h^{(k)}(z,w)$ for a weighted homogeneous holomorphic polynomial of weighted degree k and $h^{(k)}(z,\overline{z},u)$ for a weighted homogeneous polynomial of weighted degree k. We first have the following special but crucial case of the Chern-Moser normalization theorem:

Proposition 2.1 *Let $M \subset \mathbb{C}^n \times \mathbb{C}$ be a smooth Levi non-degenerate hypersurface. Let $\theta_p \in E_p$ be an appropriate real 1-form at $p \in M$. Then there is a biholomorphic map F from a neighborhood of p to a neighborhood of 0 such that $F(p) = 0$ and $F(M)$ near 0 is defined by an equation of the following normal form (up to fourth order):*

$$r = v - |z|^2_\ell + \frac{1}{4}s(z,\bar{z}) + R(z,\overline{z},u) = v - |z|^2_\ell + \frac{1}{4}\sum s^0_{\alpha\bar{\beta}\gamma\bar{\delta}} z_\alpha \bar{z}_\beta z_\gamma \bar{z}_\delta + R(z,\overline{z},u) = 0. \quad (3)$$

Here $s(z,\overline{z}) = \sum s^0_{\alpha\bar{\beta}\gamma\bar{\delta}} z_\alpha \bar{z}_\beta z_\gamma \bar{z}_\delta$, $s^0_{\alpha\bar{\beta}\gamma\bar{\delta}} = s^0_{\gamma\bar{\beta}\alpha\bar{\delta}} = s^0_{\gamma\bar{\delta}\alpha\bar{\beta}}$, $\overline{s^0_{\alpha\bar{\beta}\gamma\bar{\delta}}} = s^0_{\beta\bar{\alpha}\delta\bar{\gamma}}$ and

$$\sum_{\alpha,\beta=1}^{n} s^0_{\alpha\bar{\beta}\gamma\bar{\delta}} g_0^{\bar{\beta}\alpha} = 0 \quad (4)$$

where $g_0^{\bar{\beta}\alpha} = 0$ for $\beta \neq \alpha$, $g_0^{\bar{\beta}\beta} = 1$ for $\beta > \ell$, $g_0^{\bar{\beta}\beta} = -1$ for $\beta \leq \ell$. Also $R(z,\overline{z},u) = o_{wt}(|(z,u)|^4) \cap o(|(z,u)|^4)$. Moreover, we have $i\partial r|_0 = (F^{-1})^*\theta_p$.

Proof of Proposition 2.1: By what we discussed above, we can assume that $p = 0$ and M near $p = 0$ is defined by an equation of the form as in (1). We first show that we can get rid of all weighted third order degree terms. For this purpose, we choose a transformation of the form $f = z + f^{(2)}(z,w)$ and $g = w + g^{(3)}(z,w)$. Suppose that $F = (f_1, \cdots, f_n, g) = (f, g)$ maps $(M, p = 0)$ to a hypersurface near 0 defined by an equation of the form as in (1) but without weighted degree 3 terms in the right hand side. Substituting F into the new equation and comparing terms of weighted degree three, we get

$$\Im \left(g^{(3)} - 2i < \overline{z}, f^{(2)} >_\ell \right) |_{w=u+i|z|_\ell} = G^{(3)}(z, \overline{z}, u)$$

where $G^{(3)}$ is a certain given real-valued polynomial of weighted degree 3 in $(z, \overline{z}, u)$. Write $G^{(3)}(z, \overline{z}, u) = \Im\{a^{(1)}(z)w + \sum_{j=1}^{n} b_j^{(2)}(z)\overline{z_j}\}$. Choosing $g^{(3)} = a^{(1)}(z)w$ and $f_j^{(2)} = \frac{i}{2} b_j^{(2)}(z)$, it then does our job.

Next, we choose a holomorphic transformation of the form $f = z + f^{(3)}(z,w)$ and $g = w + g^{(4)}(z,w)$ to simplify the weighted degree 4 terms in the defining equation of $(M, p = 0)$. Suppose that M is originally defined by

$$r = v - |z|_\ell^2 + A^{(4)}(z, \overline{z}, u) + o_{wt}(4) = 0$$

and is transformed to an equation of the form:

$$r = v - |z|_\ell^2 + N^{(4)}(z, \overline{z}, u) + o_{wt}(4) = 0.$$

substituting the map F and collecting terms of weighted degree 4, we get the equation:

$$\Im \left(g^{(4)} - 2i < \overline{z}, f^{(3)} >_\ell \right) |_{w=u+i|z|_\ell} = N^{(4)}(z, \overline{z}, u) - A^{(4)}(z, \overline{z}, u).$$

Now, we like to make $N^{(4)}$ as simple as possible by choosing F. Write

$$-A^{(4)} = \Im\{b^{(4)}(z) + b^{(2)}(z)u + b^{(0)}u^2 + \sum_{j=1}^{n} c_j^{(3)}(z)\overline{z_j} + \sum_{|\alpha|=|\beta|=2} \widetilde{c_{\alpha\overline{\beta}}} z^\alpha \overline{z^\beta}\}.$$

Let

$$X^{(4)}(z,w) = b^{(4)}(z) + b^{(2)}(z)w + b^{(0)}w^2, \ -2i\delta_{j\ell} Y_j^{(3)}(z,w) = c_j^{(3)}(z) - ib^{(2)}(z)z_j - 2ib^{(0)}z_j w,$$

$$Y^{(3)} = (Y_1^{(3)}, \cdots, Y_n^{(3)}),$$

where $\delta_{j\ell}$ is 1 for $j > \ell$ and is -1 otherwise. Then $\Im \left(Y^{(4)} - 2i < \overline{z}, X^{(3)} >_\ell \right) + A^{(4)}(z, \overline{z}, u) = -\Im(b^{(0)})|z|_\ell^4 + \sum_{|\alpha|=|\beta|=2} d_{\alpha\overline{\beta}} z^\alpha \overline{z^\beta}$. By the Fischer decomposition theorem ([SW]), write in the unique way

$$-\Im(b^{(0)})|z|_\ell^4 + \sum_{|\alpha|=|\beta|=2} d_{\alpha\overline{\beta}} z^\alpha \overline{z^\beta} = h^{(2)}(z, \overline{z})|z|_\ell + h^{(4)}(z, \overline{z}).$$

Here $h^{(2)}(z,\overline{z})$ and $h^{(4)}(z,\overline{z})$ are real-valued, bi-homogeneous in $(z,\overline{z})$ and $\Delta_\ell h^{(4)}(z,\overline{z})=0$. Here, we write $\triangle_\ell = -\sum_{j\le \ell}\frac{\partial^2}{\partial z_j\partial \bar{z}_j}+\sum_{j=\ell+1}^n \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$. Notice that $h^{(2)}$ has no harmonic terms, we can find $Z^{(1)}(z)$ such that $\Re(<\overline{z}, Z^{(1)}(z)>_\ell)=0$ and $\Im(2<\overline{z},Z^{(1)}>)=h^{(2)}(z,\overline{z})$. Finally, if we define $f=z+X^{(4)}(z,w)+Z^{(1)}(z)w$ and $g^{(4)}=w+Y^{(4)}$, then (f,g) maps $(M,0)$ to a hypersurface with $R(z,\overline{z},u)=o_{wt}(4)\cap O(|(z,u)|^3)$. Now suppose that the terms with non-weighted degree of 3 or 4 in R are uniquely written as $ub^{(3)}(z,\overline{z})+u^2\Im(b^{(1)}(z))+b^{(0)}u^3+c^{(0)}u^4$ with $b^{(3)}(z,\overline{z})=\Im(c^{(3)}(z)+\sum_{|\alpha|=2,|\beta|=1}d_{\alpha\overline{\beta}}z^\alpha\overline{z}^\beta)$. Then we need to make further change of variables as follows to make $R=o_{wt}(4)\cap o(|(z,u)|^4)$ without changing $N^{(4)}(z,\overline{z})$:

$$w'=w+wc^{(3)}(z)+w^2b^{(1)}(z)+ib^{(0)}w^3+ic^{(0)}w^4,$$

$$z_j'=z_j+\delta_{j,\ell}wb^{(1)}(z)z_j+\frac{i}{2}\sum_{|\alpha|=2}wd_{\alpha,\overline{j}}z^\alpha+\delta_{j,\ell}\frac{3i}{2}w^2z_jb^{(0)}.$$

Now, the trace-free condition in (4) is equivalent to the following condition :

$$\triangle_\ell s(z,\bar{z})\equiv 0.$$

Indeed, this follows from the following fact: Let $\Delta_H=\sum_{l,k=1}^n h^{l\overline{k}}\partial_l\overline{\partial}_k$ with $\overline{h^{l\overline{k}}}=h^{k\overline{l}}$ for any l,k. Then

$$\Delta_H s^0(z,\overline{z})=4\sum_{\gamma,\delta=1}^n\sum_{\alpha,\beta=1}^n h^{\alpha\overline{\beta}}s^0_{\alpha\overline{\beta}\gamma\overline{\delta}}z_\gamma\overline{z_\delta}. \tag{5}$$

This proves the proposition. ■

We assume the notation and conclusion in Proposition 2.1. The Chern-Moser-Weyl tensor at p associated with the appropriate 1-form θ_p is defined as the 4th order tensor S_{θ_p} acting over $T_p^{(1,0)}M\otimes T_p^{(0,1)}M\otimes T_p^{(1,0)}M\otimes T_p^{(0,1)}M$. More precisely, for each $X_p,Y_p.Z_p,W_p\in T_p^{(1,0)}M$, we have the following definition:

Let F be the biholomorphic map sending M near p to the normal form as in Proposition 2.1 with $F(p)=0$, and write $F_*(X_p)=\sum_{j=1}^n a^j\frac{\partial}{\partial z_j}|_0:=X_p^0$, $F_*(Y_p)=\sum_{j=1}^n b^j\frac{\partial}{\partial z_j}|_0:=Y_p^0$, $F_*(Z_p)=\sum_{j=1}^n c^j\frac{\partial}{\partial z_j}|_0:=Z_p^0$, and $F_*(W_p)=\sum_{j=1}^n d^j\frac{\partial}{\partial z_j}|_0:=W_p^0$. Then

$$S_{\theta_p}(X_p,\overline{Y_p},Z_p,\overline{W_p}):=\sum_{\alpha,\beta,\gamma,\delta=1}^n s^0_{\alpha\overline{\beta}\gamma\overline{\delta}}a^\alpha\overline{b^\beta}c^\gamma\overline{d^\delta},\quad \text{which is denoted by } S_{\theta_0}(X_p^0,\overline{Y_p^0},Z_p^0,\overline{W_p^0}). \tag{6}$$

Since the normalization map F is not unique, we have to verify that the tensor S_{θ_p} is well-defined. Namely, we need to show that it is independent of the choice of the normal coordinates. We do this in the next section. For the rest of this section, we assume this fact and derive some basic properties for the tensor.

For a basis $\{X_\alpha\}_{\alpha=1}^n$ of $T_p^{(1,0)}M$ with $p \in M$, write $(S_{\theta_p})_{\alpha\bar\beta\gamma\bar\delta} = S_{\theta_p}(X_\alpha, \overline{X}_\beta, X_\gamma, \overline{X}_\delta)$. From the definition, we then have the following symmetric properties:

$$(S_{\theta_p})_{\alpha\bar\beta\gamma\bar\delta} = (S_{\theta_p})_{\gamma\bar\beta\alpha\bar\delta} = (S_{\theta_p})_{\gamma\bar\delta\alpha\bar\beta}$$
$$\overline{(S_{\theta_p})_{\alpha\bar\beta\gamma\bar\delta}} = (S_{\theta_p})_{\beta\bar\alpha\delta\bar\gamma},$$

and the following trace-free condition:

$$\sum_{\beta,\alpha=1}^{n} g^{\bar\beta\alpha}(S_{\theta_p})_{\alpha\bar\beta\gamma\bar\delta} = 0. \tag{7}$$

Here

$$g_{\alpha\bar\beta} = L_{\theta|_p}(X_\alpha, X_\beta) := -i < (d\theta)|_p, X_\alpha \wedge \overline{X}_\beta > \tag{8}$$

is the Levi form of M associated with θ_p and θ is a smooth extension of θ_p as a proper contact form of M near p. Also, $(g^{\bar\beta\alpha})$ is the inverse matrix of $(g_{\alpha\bar\beta})$. In the following, we write $\widetilde{\theta} = (F^{-1})^*(\theta)$.

To see the trace-free property in (7), we write that $F_*(X_\alpha) = \sum_{k=1}^n a_\alpha^k \frac{\partial}{\partial z_k}|_0$. Then $g_{\alpha\bar\beta} = L_{\theta_p}(X_\alpha, X_\beta) = -i < (d\theta)|_p, X_\alpha \wedge \overline{X}_\beta >= -i < (dF^*(\widetilde{\theta})|_p, X_\alpha \wedge \overline{X}_\beta >= -i < (i\overline{\partial}\partial r|_0, F_*(X_\alpha) \wedge \overline{F_*(X_\beta)} >= (g_0)_{k\bar l} a_\alpha^k \overline{a_\beta^l}$. Here $(g_0)_{k\bar l}$ is defined as before. Write $G = (g_{\alpha\beta}), G^0 = (g_0)_{\alpha\beta}, A = (a_k^l), B = A^{-1} := (b_k^l)$. Then we have the matrix relation: $G = AG^0\overline{A}^t$. Thus $G^{-1} = (\overline{A^t})^{-1}(G^0)^{-1}A^{-1}$, from which we have $g^{\gamma\bar\beta} = \overline{b_l^\beta}(g_0)^{j\bar l}b_j^\gamma$. Thus,

$$g^{\alpha\bar\beta}S_{\alpha\bar\beta\gamma\bar\delta} = \overline{b_l^\beta}(g_0)^{j\bar l}b_j^\alpha s^0_{\widetilde{k}\widetilde{\bar j}\widetilde{l}\widetilde{\bar m}} a_\alpha^{\widetilde{k}} \overline{a_\beta^{\widetilde{j}}} a_\gamma^{\widetilde{l}} \overline{a_\delta^{\widetilde{m}}} = (g_0)^{j\bar l} s^0_{j\bar l\widetilde{l}\widetilde{\bar m}} a_\gamma^{\widetilde{l}} \overline{a_\delta^{\widetilde{m}}} = 0.$$

We should mention the above argument can also be easily adapted to show the biholomorphic invariance of the appropriateness. Namely, if F is a CR diffeomorphism between two Levi non-degenerate hypersurfaces M and $\widetilde{M}$ of signature ℓ. For $\widetilde{\theta}_q$ is an appropriate contact 1-form at $q \in \widetilde{M}$, then $F^*(\widetilde{\theta}_q)$ is also an appropriate contact 1-form at $F^{-1}(q) \in M$.

For a smooth vector field X, Y, Z, W of type $(1, 0)$ and an appropriate smooth contact form along M, $\mathcal{S}_\theta(X, \overline{Y}, Z, \overline{W})$ is also a smooth function along M. One easy way to see this is to use the Webster-Chern-Moser-Weyl formula obtained in [We1] through the curvature tensor of the Webster pseudo-Hermitian metric, whose constructions are done by only applying the algebraic and differentiation operations on the defining function of M. Another more direct way is to trace the dependence of the tensor on the base points under the above normalization procedure.

Assume that $\ell > 0$ and define

$$\mathcal{C}_\ell = \{z \in \mathbb{C}^n : |z|_\ell = 0\}.$$

Then $\mathcal{C}_\ell$ is a real algebraic variety of real codimension 1 in $\mathbb{C}^n$ with the only singularity at 0. For each $p \in M$, write $\mathcal{C}_\ell T_p^{(1,0)} M = \{v_p \in T_p^{(1,0)} M : < (d\theta)|_p, v_p \wedge \bar{v}_p >= 0\}$. Apparently, $\mathcal{C}_l T_p^{(1,0)} M$ is independent of the choice of θ_p. Let F be a CR diffeomorphism from M to M'. We also have $F_*(\mathcal{C}_\ell T_p^{(1,0)} M) = C_\ell T_{F(p)}^{(1,0)} M'$. Write $\mathcal{C}_\ell T^{(1,0)} M = \coprod_{p\in M} \mathcal{C}_\ell T_p^{(1,0)} M$ with the natural projection π to M. We say that X is a smooth section of $\mathcal{C}_\ell T^{(1,0)} M$ if X is a smooth vector field of type $(1,0)$ along M such that $X|_p \in \mathcal{C}_\ell T_p^{(1,0)} M$ for each $p \in M$. $\mathcal{C}_\ell T^{(1,0)} M$ is a kind of smooth bundle with each fiber isomorphic to $\mathcal{C}_\ell$.

$\mathcal{C}_\ell$ is obviously a uniqueness set for holomorphic functions. The following lemma shows that it is also a uniqueness set for the Chern-Moser-Weyl curvature tensor. (For the proof, see Lemma 2.1 of [HZh].)

Proposition 2.2 *(Huang-Zhang [HZh]) (I). Suppose that $H(z,\bar{z})$ is a real real-analytic function in $(z,\bar{z})$ near 0. Assume that $\triangle_\ell H(z,\bar{z}) \equiv 0$ and $H(z,\bar{z})|_{\mathcal{C}_\ell} = 0$. Then $H(z,\bar{z}) \equiv 0$ near 0. (II). Assume the above notation and $\ell > 0$. If $S_{\theta_p}(X,\overline{X},X,\overline{X}) = 0$ for any $X \in \mathcal{C}_\ell T_p^{(1,0)} M$, then $S_{\theta|_p} \equiv 0$.*

3 Transformation law for the Chern-Moser-Weyl tensor

We next show that the Chern-Moser-Weyl tensor defined in the previous section is well-defined by proving a transformation law. We follow the approach and expositions developed in Huang-Zhang [HZh].

Let $\widetilde{M} \subset \mathbb{C}^{N+1} = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}\}$ be also a Levi non-degenerate real hypersurface near 0 of signature $\ell \geq 0$ defined by an equation of the form:

$$\widetilde{r} = \Im \widetilde{w} - |\widetilde{z}|_\ell^2 + o(|\widetilde{z}|^2 + |\widetilde{z}\widetilde{u}|) = 0. \qquad (9)$$

Let $F := (f_1, \ldots, f_n, \phi, g) : M \to \widetilde{M}$ be a smooth CR diffeomorphism. Then, as in [Hu1] and [BH], we can write

$$\begin{aligned} \tilde{z} &= \tilde{f}(z,w) = (f_1(z,w), \ldots, f_n(z,w)) = \lambda z U + \vec{a} w + O(|(z,w)|^2) \\ \tilde{w} &= g(z,w) = \sigma \lambda^2 w + O(|(z,w)|^2). \end{aligned} \tag{10}$$

Here $U \in SU(n,\ell)$. (Namely $< XU, Y\overline{U} >_\ell = < X, Y >_\ell$ for any $X, Y \in \mathbb{C}^n$). Moreover, $\vec{a} \in \mathbb{C}^n$, $\lambda > 0$ and $\sigma = \pm 1$ with $\sigma = 1$ for $\ell < \frac{n}{2}$. When $\sigma = -1$, by considering $F \circ \tau_{n/2}$ instead of F, where $\tau_{\frac{n}{2}}(z_1, \ldots, z_{\frac{n}{2}}, z_{\frac{n}{2}+1}, \ldots, z_n, w) = (z_{\frac{n}{2}+1}, \ldots, z_n, z_1, \ldots, z_{\frac{n}{2}}, -w)$, we can make $\sigma = 1$. Hence, we will assume in what follows that $\sigma = 1$.

Write $r_0 = \frac{1}{2}\Re\{g''_{ww}(0)\}$, $q(\tilde{z}, \tilde{w}) = 1 + 2i < \tilde{z}, \lambda^{-2}\overline{\vec{a}} >_\ell + \lambda^{-4}(r_0 - i|\vec{a}|^2_\ell)\tilde{w}$,

$$T(\tilde{z}, \tilde{w}) = \frac{(\lambda^{-1}(\tilde{z} - \lambda^{-2}\vec{a}\tilde{w})U^{-1}, \lambda^{-2}\tilde{w})}{q(\tilde{z}, \tilde{w})}. \tag{11}$$

Then

$$F^\sharp(z,w) = (\tilde{f}^\sharp, g^\sharp)(z,w) := T \circ F(z,w) = (z,w) + O(|(z,w)|^2) \tag{12}$$

with $\Re\{g^{\sharp''}_{ww}(0)\} = 0$.

Assume that $\widetilde{M}$ is also defined in the Chern-Moser normal form up to the 4th order:

$$\tilde{r} = \Im\tilde{w} - |\tilde{z}|^2_\ell + \frac{1}{4}\tilde{s}(\tilde{z}, \bar{\tilde{z}}) + o_{wt}(|(\tilde{z}, \widetilde{u})|^4) = 0. \tag{13}$$

Then $M^\sharp = T(\widetilde{M})$ is defined by

$$r^\sharp = \Im w^\sharp - |z^\sharp|^2_\ell + \frac{1}{4}s^\sharp(z^\sharp, \bar{z^\sharp}) + o_{wt}(|(z^\sharp, w^\sharp)|^4) = 0 \tag{14}$$

with $s^\sharp(z^\sharp, \bar{z^\sharp}) = \lambda^{-2}\tilde{s}(\lambda z^\sharp U, \lambda\overline{z^\sharp U})$.

One can verify that

$$\Big(-\sum_{j=1}^{\ell} \frac{\partial^2}{\partial z^\sharp_j \partial \bar{z}^\sharp_j} + \sum_{j=\ell+1}^{N} \frac{\partial^2}{\partial z^\sharp_j \partial \bar{z}^\sharp_j}\Big) s^\sharp(z^\sharp, \overline{z^\sharp}) = 0. \tag{15}$$

Therefore (14) is also in the Chern-Moser normal form up to the 4th order. Write $F^\sharp(z,w) = \sum_{k=1}^\infty F^{\sharp(k)}(z,w)$. Since $F^\sharp$ maps M into $M^\sharp = T(\widetilde{M})$, we get the following

$$\begin{aligned} &\Im\{\sum_{k\geq 2} g^{\sharp(k+1)}(z,w) - 2i\sum_{k\geq 2} < f^{\sharp(k)}(z,w), \bar{z} >_\ell\} \\ &\qquad = \sum_{k_1,\ k_2 \geq 2} < f^{\sharp(k_1)}(z,w), \overline{f^{\sharp(k_2)}(z,w)} >_\ell + \frac{1}{4}(s(z,\bar{z}) - s^\sharp(z,\overline{z})) + o_{wt}(4) \end{aligned} \tag{16}$$

over $\Im w = |z|_\ell^2$. Here, we write $F^\sharp(z,w) = (f^\sharp(z,w), g^\sharp(z,w))$.
Collecting terms of weighted degree 3 in (16), we get

$$\Im\{g^{\sharp(3)}(z,w) - 2i < f^{\sharp(2)}(z,w), \bar{z} >_\ell\} = 0 \ \text{ on } \ \Im w = |z|_\ell^2.$$

By [Hu1], we get $g^{\sharp(3)} \equiv 0, f^{\sharp(2)} \equiv 0$.
Collecting terms of weighted degree 4 in (16), we get

$$\Im\{g^{\sharp(4)}(z,w) - 2i < f^{\sharp(3)}(z,w), \bar{z} >_\ell\} = \frac{1}{4}(s(z,\bar{z}) - s^\sharp(z,\overline{z})).$$

Similar to the argument in [Hu1] and making use of the fact that $\Re\{\frac{\partial^2 g^{\sharp(4)}}{\partial w^2}(0)\} = 0$, we get the following:

$$\begin{aligned} g^{\sharp(4)} \equiv 0, \ f^{\sharp(3)}(z,w) = \frac{i}{2}a^{(1)}(z)w, \\ < a^{(1)}(z), \bar{z} >_\ell |z|_\ell^2 = \frac{1}{4}(s(z,\bar{z}) - s^\sharp(z,\overline{z})) = \frac{1}{4}(s(z,\bar{z}) - \lambda^{-2}\widetilde{s}(\lambda z U, \overline{\lambda z U})). \end{aligned} \tag{17}$$

Since the right hand side of the above equation is annihilated by Δ_ℓ and the left hand side of the above equation is divisible by $|z|_\ell^2$. We conclude that $f^{\sharp(3)}(z,w) = 0$ and

$$s(z,\bar{z}) = \lambda^{-2}\widetilde{s}(\lambda z U, \overline{\lambda z U}). \tag{18}$$

Write $\theta_0 = i\partial r|_0$ and $\widetilde{\theta_0} = i\partial\widetilde{r}|_0$. Then $F^*(\widetilde{\theta_0}) = \lambda^2\theta_0$. For any $X = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}|_0$, $F_*(X) = \lambda(z_1\frac{\partial}{\partial \widetilde{z_1}}|_0, \cdots, z_n\frac{\partial}{\partial \widetilde{z_n}}|_0)U$. Under this notation, (19) can be written as

$$S^0_{F^*(\widetilde{\theta_0})}(X, \overline{X}, X, \overline{X}) = S^0_{\widetilde{\theta_0}}(F_*(X), \overline{F_*(X)}, F_*(X), \overline{F_*(X)}).$$

This immediately gives the following transformation law and thus the following theorem, too.

$$S^0_{F^*(\widetilde{\theta_0})}(X, \overline{Y}, Z, \overline{W}) = S^0_{\widetilde{\theta_0}}(F_*(X), \overline{F_*(Y)}, F_*(Z), \overline{F_*(W)}), \ \text{for } X, Y, Z, W \in T_0^{(1,0)}M. \tag{19}$$

Theorem 3.1 *(1). The Chern-Moser-Weyl tensor defined in the previous section is independent of the choice of the normal coordinates and thus is a well-defined fourth order tensor. (2). Let F be a CR diffeomorphism between two Levi non-degenerate hypersurfaces $M, M' \subset \mathbb{C}^{n+1}$. Suppose $F(p) = q$. Then, for any appropriate contact 1-form $\widetilde{\theta}_q$ of $\widetilde{M}$ at q*

and a vector $v \in T_p^{(1,0)}M$, we have the following transformation formula for the corresponding Chern-Moser-Weyl tensor:

$$\tilde{S}_{\tilde{\theta}_p}(F_*(v_1), \overline{F_*(v_2)}, F_*(v_3), \overline{F_*(v_4)}) = S_{F^*(\tilde{\theta}_q)}(v_1, \overline{v_2}, v_3, \overline{v_4}). \tag{20}$$

Proof: Let θ_p be an appropriate contact form of M at p, and let F_1, F_2 be two normalization (up to fourth order) of M at p. Suppose that $F_1(M)$ and $F_2(M)$ are defined near 0 by equations $r_1 = 0$ and $r_2 = 0$ as in (1), respectively. Write $\Phi = F_2 \circ F_1^{-1}$ and $\theta_0^1 = i\partial r_1$, $\theta_0^2 = i\partial r_2$. We also assume that $F_1^*(\theta_0^1) = \theta_p$ and $F_2^*(\theta_0^2) = \theta_p$. Then for any $X_p, Y_p, Z_p, W_p \in T_p^{(1,0)}M$, we have

$$S^1_{\theta_p}(X_p, \overline{Y_p}, Z_p, \overline{W_p}) = S^1_{\theta_0^1}((F_1)_*(X_p), \overline{(F_1)_*(Y_p)}, (F_1)_*(Z_p), \overline{(F_1)_*(W_p)})$$

if we define the tensor at p by applying F_2. We also have

$$S^2_{\theta_p}(X_p, \overline{Y_p}, Z_p, \overline{W_p}) = S^2_{\theta_0^2}((F_2)_*(X_p), \overline{(F_2)_*(Y_p)}, (F_2)_*(Z_p), \overline{(F_2)_*(W_p)}),$$

if we define the tensor at p by applying F_2. Since $\theta_0^2 = \Phi^*(\theta_0^1)$, and $\Phi_*((F_1)_*(X_p)) = (F_2)_*(X_p)$, by the transformation law obtained in (19), we see the proof in Part I of the theorem. The proof in Part II of the theorem also follows easily from the formula in (19).

4 A monotonicity theorem for the Chern-Moser-Weyl tensor

We now let $M_\ell \subset \mathbb{C}^{n+1}$ be a Levi non-degenerate hypersurface with signature $\ell > 0$ defined in the normal form as in (3). Let $F = (f_1, \cdots, f_N, g)$ be a CR-transversal CR embedding from M_ℓ into $\mathbb{H}_\ell^{N+1}$ with $N \geq n$. Then again as in Section 3, a simple linear algebra argument ([HZh]) shows that after a holomorphic change of variables, we can make F into the following preliminary normal form:

$$\begin{aligned} \tilde{z} &= \tilde{f}(z,w) = (f_1(z,w), \ldots, f_N(z,w)) = \lambda z U + \vec{a} w + O(|(z,w)|^2) \\ \tilde{w} &= g(z,w) = \sigma \lambda^2 w + O(|(z,w)|^2). \end{aligned} \tag{21}$$

Here U can be extended to an $N \times N$ matrix $\widetilde{U} \in SU(N, \ell)$. Moreover, $\vec{a} \in \mathbb{C}^N$, $\lambda > 0$ and $\sigma = \pm 1$ with $\sigma = 1$ for $\ell < \frac{n}{2}$. When $\sigma = -1$, qs discussed before, by considering $F \circ \tau_{n/2}$

instead of F, where $\tau_{\frac{n}{2}}(z_1,\ldots,z_{\frac{n}{2}},z_{\frac{n}{2}+1},\ldots,z_n,w)=(z_{\frac{n}{2}+1},\ldots,z_n,z_1,\ldots,z_{\frac{n}{2}},-w)$, we can make $\sigma=1$. Hence, we will assume that $\sigma=1$.

Write $r_0=\frac{1}{2}\Re\{g''_{ww}(0)\}$, $q(\tilde{z},\tilde{w})=1+2i<\tilde{z},\lambda^{-2}\overline{\vec{a}}>_\ell+\lambda^{-4}(r_0-i|\vec{a}|^2_\ell)\tilde{w}$,

$$T(\tilde{z},\tilde{w})=\frac{(\lambda^{-1}(\tilde{z}-\lambda^{-2}\vec{a}\tilde{w})\widetilde{U}^{-1},\lambda^{-2}\tilde{w})}{q(\tilde{z},\tilde{w})}. \tag{22}$$

Then

$$F^\sharp(z,w)=(\tilde{f}^\sharp,g^\sharp)(z,w):=T\circ F(z,w)=(z,0,w)+O(|(z,w)|^2) \tag{23}$$

with $\Re\{g^{\sharp''}_{ww}(0)\}=0$. Now, $T(\mathbb{H}^{N+1}_\ell)=\mathbb{H}^{N+1}_\ell$. With the same argument as in the previous section, we also arrive at the following:

$$\begin{gathered} g^{\sharp(3)}=g^{\sharp(4)}\equiv 0,\ f^{\sharp(3)}(z,w)=\frac{i}{2}a^{(1)}(z)w, \\ <a^{(1)}(z),\bar{z}>_\ell|z|^2_\ell=|\phi^{\sharp(2)}(z)|^2+\frac{1}{4}s(z,\bar{z}). \end{gathered} \tag{24}$$

In the above equation, if we let z be such that $|z|_\ell=0$, we see that $s(z,\overline{z})\leq 0$. Now, if F is not CR transversal but not totally non-degenerate in the sense that F does not map an open subset of $\mathbb{C}^n$ into $\mathbb{H}^N_\ell$ (see [HZh]), then one can apply this result on a dense open subset of M [BER] where F is CR transversal and then take a limit as did in [HZh]. Then we have the following special case of the monotonicity theorem for the Chern-Moser-Weyl tensor obtained in Huang-Zhang [HZh]:

Theorem 4.1 *([HZh]) Let $M_\ell\subset\mathbb{C}^{n+1}$ be a Levi non-degenerate real hypersurface of signature ℓ. Suppose that F is a holomorphic mapping defined in a (connected) open neighborhood U of M in $\mathbf{C}^{n+1}$ that sends M_ℓ into $\mathbf{H}^{N+1}_\ell\subset\mathbb{C}^{N+1}$. Assume that $F(U)\not\subset\mathbf{H}^{N+1}_\ell$. Then when $\ell<\frac{n}{2}$, the Chern-Moser-Weyl curvature tensor with respect to any appropriate contact form θ is pseudo semi-negative in the sense that for any $p\in M$, the following holds:*

$$\mathcal{S}_{\theta|_p}(v_p,\overline{v_p},v_p,\overline{v_p})\leq 0,\quad for\ v_p\in\mathcal{C}_\ell T^{(1,0)}_p M. \tag{25}$$

When $\ell=\frac{n}{2}$, along a certain contact form θ, $\mathcal{S}_\theta$ is pseudo negative.

5 Counter-examples to the embeddability problem for compact algebraic Levi non-degenerate hypersurfaces with positive signature into hyperquadrics

In this section, we apply Theorem 4.1 to construct a compact Levi-nondegenerate hypersurface in a projective space, for which any piece of it can not be holomorphically embedded into a hyperquadric of any dimension with the same signature. This section is based on the work in the last section of Huang-Zaitsev [HZa].

Let n, ℓ be two integers with $1 < \ell \le n/2$. For any ϵ, define

$$M_\epsilon := \left\{ [z_0, \cdots, z_{n+1}] \in \mathbb{P}^{n+1} : |z|^2 \left(-\sum_{j=0}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n+1} |z_j|^2 \right) + \epsilon \left(|z_1|^4 - |z_{n+1}|^4 \right) = 0 \right\}.$$

Here $|z|^2 = \sum_{j=0}^{n+1} |z_j|^2$ as usual. For $\epsilon = 0$, M_ϵ reduces to the generalized sphere with signature ℓ, which is the boundary of the generalized ball

$$\mathbb{B}_\ell^{n+1} := \left\{ \{[z_0, \cdots, z_{n+1}] \in \mathbb{P}^{n+1} : -\sum_{j=0}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n+1} |z_j|^2 < 0 \right\}.$$

The boundary $\partial\mathbb{B}_\ell^{n+1}$ is locally holomorphically equivalent to the hyperquadric $\mathbb{H}_\ell^{n+1} \subset \mathbb{C}^{n+1}$ of signature ℓ defined by $\Im z_{n+1} = -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n+1} |z_j|^2$, where $(z_1, \cdots, z_{n+1})$ is the coordinates of $\mathbb{C}^{n+1}$.

For $0 < \epsilon << 1$, M_ϵ is a compact smooth real-algebraic hypersurface with Levi form non-degenerate of the same signature ℓ.

Theorem 5.1 *([HZa]) There is an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, the following holds: (i) M_ϵ is a smooth real-algebraic hypersurface in $\mathbb{P}^{n+1}$ with non-degenerate Levi form of signature ℓ at every point. (ii) There does not exist any holomorphic embedding from any open piece of M_ℓ into $\mathbb{H}_\ell^{N+1}$.*

When $0 < \epsilon << 1$, since M_ϵ is a small algebraic deformation of the generalzied sphere, we see that M_ϵ must also be a compact real-algebraic Levi non-degenerate hypersurface in $\mathbb{P}^{n+1}$ with signature ℓ diffeomorphic to the generalized sphere which is the boundary of the generalized ball $\mathbb{B}_\ell^{n+1} \subset \mathbb{P}^{n+1}$.

Proof of Theorem 5.1: The proof uses the following algebraicity of the first author:

Theorem 5.2 (Hu2, Corollary in §2.3.5) *Let $M_1 \subset \mathbb{C}^n$ and $M_2 \subset \mathbb{C}^N$ with $N \ge n \ge 2$ be two Levi non-degenerate real-algebraic hypersurfaces. Let $p \in M_1$ and U_p be a small connected open neighborhood of p in $\mathbf{C}^n$ and F be a holomorphic map from U_p into $\mathbf{C}^N$ such that $F(U_p \cap M_1) \subset M_2$ and $F(U_p) \not\subset M_2$. Suppose that M_1 and M_2 have the same signature ℓ at p and $F(p)$, respectively. Then F is algebraic in the sense that each component of F satisfies a nontrivial holomorphic polynomial equation.*

Next, we compute the Chern-Moser-Weyl tensor of M_ϵ at the point

$$P_0 := [\xi_0^0, \cdots, \xi_{n+1}^0], \quad \xi_j^0 = 0 \text{ for } j \neq 0, \ell+1, \quad \xi_0^0 = 1, \quad \xi_{\ell+1}^0 = 1,$$

and consider the coordinates

$$\xi_0 = 1, \quad \xi_j = \frac{\eta_j}{1+\sigma}, \quad j = 1, \cdots, \ell, \quad \xi_{\ell+1} = \frac{1-\sigma}{1+\sigma}, \quad \xi_{j+1} = \frac{\eta_j}{1+\sigma}, \quad j = \ell+1, \cdots, n.$$

Then in the (η, σ)-coordinates, P_0 becomes the origin and M_ϵ is defined near the origin by an equation in the form:

$$\rho = -4\Re\sigma - \sum_{j=1}^{\ell} |\eta_j|^2 + \sum_{j=\ell+1}^{n} |\eta_j|^2 + a(|\eta_1|^4 - |\eta_n|^4) + o(|\eta|^4) = 0, \tag{26}$$

for some $a > 0$. Now, let $Q(\eta, \overline{\eta}) = -a(|\eta_1|^4 - |\eta_n|^4)$ and make a standard ℓ-harmonic decomposition [SW]:

$$Q(\eta, \overline{\eta}) = N^{(2,2)}(\eta, \overline{\eta}) + A^{(1,1)}(\eta, \overline{\eta})|\eta|_\ell^2. \tag{27}$$

Here $N^{(2,2)}(\eta, \eta)$ is a $(2,2)$-homogeneous polynomial in $(\eta, \overline{\eta})$ such that $\Delta_\ell N^{(2,2)}(\eta, \overline{\eta}) = 0$ with Δ_ℓ as before. Now $N^{(2,2)}$ is the Chern-Moser-Weyl tensor of M_ϵ at 0 (with respect to an obvious contact form) with $N^{(2,2)}(\eta, \overline{\eta}) = Q(\eta, \overline{\eta})$ for any $\eta \in \mathcal{C}T_0^{(1,0)} M_e$. Now the value of the Chern-Moser-Weyl tension has negative and positive value at $X_1 = \frac{\partial}{\partial \eta_1} + \frac{\partial}{\partial \eta_{\ell+1}}|_0$ and $X_2 = \frac{\partial}{\partial \eta_2} + \frac{\partial}{\partial \eta_n}|_0$, respectively. If $\ell > 1$, then both X_1 and X_2 are in $\mathcal{C}T_0^{(1,0)} M_e$. We see that the Chern-Moser-Weyl tensor can not be pseudo semi-definite near the origin in such a coordinate system.

Next, suppose an open piece U of M_ϵ can be holomorphically and transversally embedded into the $\mathbf{H}_\ell^{N+1}$ for $N > n$ by F. Then by the algebraicity result in Theorem 5.2, F is algebraic. Since the branching points of F and the points where F is not defined (poles or points of indeterminancy of F) are contained in a complex-algebraic variety of codimension at most one, F extends holomorphically along a smooth curve γ starting from some point in U and ending up at some point $p^*(\approx 0) \in M_\ell$ in the (η, σ)-space where the Chern-Moser-Weyl tensor of M_ϵ is not pseudo-semi-definite. By the uniqueness of real-analytic functions, the extension of F must also map an open piece of p^* into $\mathbf{H}_\ell^{N+1}$. The extension is not totally degenerate. By Theorem 4.1, we get a contradiction. ■

6 Non-embeddability of compact strongly psuedo-convex real algebraic hypersurfaces into spheres

As discussed in the previous sections, spheres serve as the model of strongly pseudoconvex real hypersurfaces where the Chern-Moser-Weyl tensor vanishes. An immediate application of the invariant property for the Chern-Moser-Weyl tensor is that very rare strongly pseudoconvex real hypersurfaces can be biholomorphically mapped to a unit sphere. Motivated by various embedding theorems in geometries (Nash embedding, Remmert embedding theorems, etc), a natural question to pursue in Several Complex Variables is to determine when a real hypersurface in $\mathbb{C}^n$ can be holomorphically embedded into the unit sphere $\mathbb{S}^{2N-1} = \{Z \in \mathbb{C}^N : ||Z||^2 = 1\}$.

By a holomorphic embedding of $M \subset \mathbb{C}^n$ into $M' \subset \mathbb{C}^N$, we mean a holomorphic embedding of an open neighborhood U of M into a neighborhood U' of M', sending M into M'. We also say M is locally holomorphically embeddable into M' at $p \in M$, if there is a neighborhood V of p and a holomorphic embedding $F : V \to \mathbb{C}^N$ sending $M \cap V$ into M'.

A real hypersurface holomorphically embeddable into a sphere is necessarily strongly pseudoconvex and real-analytic. However, due to results by Forstnerić [For1] (See a recent work [For2] for further result) and Faran [Fa], not every strongly pseudoconvex real-analytic hypersurface can be embedded into a sphere. Explicit examples of non-embeddable strongly pseudoconvex real-analytic hypersurfaces constructed much later in [Za1]. Despite a vast of literature devoted to the embeddability problem, the following question remains an open question of long standing. Here recall a smooth real hypersurface in an open subset U of $\mathbb{C}^n$ is called real-algebraic, if it has a real-valued polynomial defining function.

Question 6.1 *Is every compact real-algebraic strongly pseudoconvex real hypersuraface in $\mathbb{C}^n$ holomorphically embeddable into a sphere of sufficiently large dimension?*

Part of the motivation to study this embeddability problem is a well-known result due to Webster [We2] which states that every real-algebraic Levi-nondegenerate hypersurface admits a transversal holomorphic embedding into a non-degenerate hyperquadric in sufficiently large complex space. (See also [KX] for further study along this line.) Notice that in [HZa], the authors showed that there are many compact real-algebraic pseudoconvex real hypersurfaces with just one weakly pseudoconvex point satisfying the following property: Any open piece of them cannot be holomorphically embedded into any compact real-algebraic strongly pseudoconvex hypersurfaces which, in particular, includes spheres. Many other related results can be found in the work of Ebenfelt-Son [ES], Fornaess [Forn], etc.

In [HLX], the authors constructed the following family of compact real-algebraic strongly pseudoconvex real hypersurfaces:

$$M_\epsilon = \{(z, w) \in \mathbb{C}^2 : \varepsilon_0(|z|^8 + c\mathrm{Re}|z|^2 z^6) + |w|^2 + |z|^{10} + \epsilon|z|^2 - 1 = 0\}, \quad 0 < \epsilon < 1. \quad (28)$$

Here, $2 < c < \frac{16}{7}$, $\varepsilon_0 > 0$ is a sufficiently small number such that M_ε is smooth for all $0 \le \epsilon < 1$. An easy computation shows that for any $0 < \epsilon < 1, M_\epsilon$ is strongly pseudoconvex. M_ϵ is indeed a small algebraic deformation of the boundary of the famous Kohn-Nirenberg domain [KN]. It is shown in [HLX] that for any integer N, there exists a small number $0 < \epsilon(N) < 1$, such that for any $0 < \epsilon < \epsilon(N)$, M_ϵ cannot be locally holomorphically embedded into the unit sphere $\mathbb{S}^{2N-1}$ in $\mathbb{C}^N$. More precisely, any holomorphic map sending an open piece of M_ϵ to $\mathbb{S}^{2N-1}$ must be a constant map. We will write

$$\rho_\epsilon = \rho_\epsilon(z, w, \overline{z}, \overline{w}) := \varepsilon_0(|z|^8 + c\mathrm{Re}|z|^2 z^6) + |w|^2 + |z|^{10} + \epsilon|z|^2 - 1.$$

We first fix some notations. Let $M \subset \mathbb{C}^n$ be a real-algebraic subset defined by a family of real-valued polynomials $\{\rho_\alpha(Z, \overline{Z}) = 0\}$, where Z is the coordinates of $\mathbb{C}^n$. Then the complexification $\mathcal{M}$ of M is the complex-algbraic subset in $\mathbb{C}^n \times \mathbb{C}^n$ defined by $\rho_\alpha(Z, W) = 0$ for each $\alpha, (Z, W) \in \mathbb{C}^n \times \mathbb{C}^n$. Then for $p \in \mathbb{C}^n$, the Segre variety of M associated with the point p is defined by $Q_p := \{Z \in \mathbb{C}^n : (Z, \overline{p}) \in \mathcal{M}\}$. The geometry of Segre varieties of a real-analytic hypersurface has been used in many literatures since the work of Segre [S] and Webster [We].

In this note, fundamentally based on our previous joint work with Li [HLX], we show that M_ϵ cannot be locally holomorphically embedded into any unit sphere. The other important observation we need is the fact that for some $p \in M_\epsilon$, the associated Segre variety Q_p cuts M_ϵ along a one dimensional real analytic subvariety inside M_ϵ. The geometry related to intersection of the Segre variety with the boundary plays an important role in the study of many problems in Several Complex Variables. (We mention, in particular, the work of D'Angelo-Putinar [DP], Huang-Zaitsev [HZa]).

This then provides a counter-example to a long standing open question— Question 6.1. (See [HZa] for more discussions on this matter).

Theorem 6.2 *There exist compact real-algebraic strongly pseudoconvex real hypersurfaces in $\mathbb{C}^2$, diffeomorphic to the sphere, that are not locally holomorphically embeddable into any sphere. In particular, for sufficiently small positive $\varepsilon_0, \epsilon, M_\epsilon$ cannot be locally holomorphically embedded into any sphere. More precisely, a local holomorphic map sending an open piece of M_ϵ to a unit sphere must be a constant map.*

Write $D_\epsilon = \{\rho_\epsilon < 0\}$ as the interior domain enclosed by M_ϵ. Since M_ϵ is a small smooth deformation of $\{|z|^{10} + |w|^2 = 1\}$ for small ε_0 and ϵ. This imples M_ϵ is diffeomorphic to the unit sphere $\mathbb{S}^3$ for sufficiently small ε_0 and ϵ. Consequently, M_ϵ separates $\mathbb{C}^2$ into two connected components D_ϵ and $\mathbb{C}^2 \setminus \overline{D_\epsilon}$.

Proposition 6.3 *Let* $p_0 = (0, 1) \in M_\epsilon$. *Let* Q_{p_0} *be the Segre variety of* M_ϵ *associated to* p_0. *There exists* $\widetilde{\epsilon} > 0$ *such that for each* $0 < \epsilon < \widetilde{\epsilon}$, $Q_{p_0} \cap M_\epsilon$ *is a real analytic subvariety of dimension one.*

Proof of Proposition 6.3: It suffices to show that there exists $q \in Q_{p_0}$ such that $q \in D_\epsilon$. Note that $Q_{p_0} = \{(z, w) : w = 1\}$. Set

$$\psi(z, \epsilon) = \varepsilon_0(|z|^8 + c\text{Re}|z|^2 z^6) + |z|^{10} + \epsilon|z|^2, \ 0 \leq \epsilon < 1.$$

Note $q = (\mu_0, 1) \in D_\epsilon$ if and only if $\psi(\mu_0, \epsilon) < 0$. Now, set $\phi(\lambda, \epsilon) = \varepsilon_0\lambda^8(1-c)+\lambda^{10}+\epsilon\lambda^2, 0 \leq \epsilon < 1$. First we note there exists small $\lambda' > 0$, such that $\phi(\lambda', 0) < 0$. Consequently, we can find $\widetilde{\epsilon} > 0$ such that for each $0 < \epsilon \leq \widetilde{\epsilon}, \phi(\lambda', \epsilon) < 0$. Write $\mu_0 = \lambda' e^{i\frac{\pi}{6}}$. It is easily to see that $\psi(\mu_0, \epsilon) < 0$ if $0 < \epsilon \leq \widetilde{\epsilon}$. This establishes Proposition 6.3. ∎

Proposition 6.4 *Let* $M := \{Z \in \mathbb{C}^n : \rho(Z, \overline{Z}) = 0\}, n \geq 2$, *be a compact, connected, strongly pseudo-convex real-algebraic hypersurface. Assume that there exists a point* $p \in M$ *such that the associated Segre variety* Q_p *of* M *is irreducible and* Q_p *intersects* M *at infinitely many points. Let* F *be a holomorphic rational map sending an open piece of* M *to the unit sphere* $\mathbb{S}^{2N-1}$ *in some* $\mathbb{C}^N$. *Then* F *is a constant map.*

Proof of Proposition 6.4: Let D be the interior domain enclosed by M. From the assumption and a theorem of Chiappari [Ch], we know F is holomorphic in a neighborhood U of $\overline{D}$ and sends M to $\mathbb{S}^{2N-1}$. Consequently, if we write $\mathcal{S}$ as the singular set of F, then it does not intersect U. Write Q'_q for the Segre variety of $\mathbb{S}^{2N-1}$ associated to $q \in \mathbb{C}^N$. We first conclude by complexification that for a small neighborhood V of p,

$$F(Q_p \cap V) \subset Q'_{F(p)}. \tag{29}$$

Note that $\mathcal{S} \cap Q_p$ is a Zariski close proper subset of Q_p. Notice that Q_p is connected as it is irreducible. We conclude by unique continuation that if $\widetilde{p} \in Q_p$ and F is holomorphic at $\widetilde{p}$, then $F(\widetilde{p}) \in Q'_{F(p)}$. In particular, if $\widetilde{p} \in Q_p \cap M$, then $F(\widetilde{p}) \in Q'_{F(p)} \cap \mathbb{S}^{2N-1} = \{F(p)\}$. That is, $F(\widetilde{p}) = F(p)$.

Notice by assumption that $Q_p \cap M$ is a compact set and contains infinitely many points. Let $\hat{p}$ be an accumulation point of $Q_p \cap M$. Clearly, by what we argued above, F is not one-to-one in any neighborhood of $\hat{p}$. This shows that F is constant. Indeed, suppose F is not a constant map. We then conclude that F is a holomorphic embedding near $\hat{p}$ by a standard Hopf lemma type argument (see [Hu2], for instance) for both M_ϵ and $\mathbb{S}^{2N-1}$ are strongly pseudo-convex. This completes the proof of Proposition 6.4. ∎

Proof of Theorem 6.2: Pick $p_0 = (0, 1) \in M_\epsilon$. Notice that the associated Segre variety $Q_{p_0} = \{(z, 1) : z \in \mathbb{C}\}$ is an irreducible complex variety in $\mathbb{C}^2$. Let ϵ, ε_0 be sufficiently small such that Proposition 6.3 holds.

Now, let F be a holomorphic map defined in a small neighborhood U of some point $q \in M_\epsilon$ that sends an open piece of M_ϵ into $\mathbb{S}^{2N-1}, N \in \mathbb{N}$. It is shown in [HLX] that F is a rational map. Then it follows from Proposition 6.4 that F is a constant map. We have thus established Theorem 6.2. ■

References

[BER] S. Baouendi, P. Ebenfelt and L. Rothschild, Transversality of holomorphic mappings between real hypersurfaces in different dimensions, Comm. Anal. Geom. 15 (2007), no. 3, 589-611.

[BH] M. S. Baouendi and X. Huang, Super-rigidity for holomorphic mappings between hyperquadrics with positive signature, J. Diff. Geom. 69 (2005), 379-398.

[Ch] S. Chiappari, Holomorphic extension of proper meromorphic mappings, Mich. Math. J. 38, 167- 174 (1991).

[CM] S. S. Chern and J. K. Moser, Real hypersurfaces in complex manifolds, Acta Math. 133 (1974), 219-271.

[DP] J. D'Angelo and M. Putinar, Hermitian complexity of real polynomial ideals, Int. J. Math., 23 (2012), 1250026.

[ES] P. Ebenfelt and D. Son, On the existence of holomorphic embeddings of strictly pseudoconvex algebraic hypersurfaces into spheres, May, 2012. (arXiv:1205.1237).

[Fa] J. J. V. Faran, The nonimbeddability of real hypersurfaces in spheres. Proc. Amer. Math. Soc. 103, 3 (1988), 902-904.

[Forn] J. E. Fornæss, Strictly pseudoconvex domains in convex domains, Amer. J. Math. 98 (1976), 529-569.

[For1] F. Forstnerić, Embedding strictly pseudoconvex domains into balls, Transactions of AMS (Vol. 295), No. 1 (May, 1986), 347-368.

[For2] F. Forstnerić, Most real analytic Cauchy-Riemann manifolds are nonalgebraizable, Manuscripta Math. 115 (2004), 489–494.

[Hu1] X. Huang, On a linearity problem of proper holomorphic mappings between balls in complex spaces of different dimensions, J. Diff. Geom. 51 (1999), 13-33.

[Hu2] X. Huang, On the mapping problem for algebraic real hypersurfaces in complex spaces of different dimensions, Annales de L'Institut Fourier 44 (1994), 433-463.

[Hu3] X. Huang, Geometric Analysis in Several Complex Variables, Ph. D. Thesis, Washington University, St. Louis, 1994. (http://www.math.rutgers.edu/ huangx/thesis-huang.pdf)

[HLX] X. Huang, X. Li and M. Xiao, Non-embeddability into a fixed sphere for a family of compact real algebraic hypersurfaces, Int Math Res Notices (16) 2015, 7382-7393.

[HZa] X. Huang and D. Zaitsev, Non-embeddable real algebraic hypersurfaces, Math. Z. 275 (2013) 657-671.

[HZh] X. Huang and Y. Zhang, Monotonicity for the Chern-Moser-Weyl curvature tensor and CR embeddings, Science in China Series A: Mathematics Dec., 2009, Vol. 52, No. 12, 2617-2627.

[KX] I. Kossovskiy and M. Xiao, On the embeddability of real hypersurfaces into hyperquadrics, submitted, arxiv: 1509. 01962.

[KN] J. J. Kohn and L. Nirenberg, A pseudo-convex domain not admitting a holomorphic support function, Math. Ann., 201 (1973), 265-268.

[S] B. Segre, Intorno al problem di Poincaré della rappresentazione pseudo-conform, Rend. Acc. Lincei 13 (1931), 676-683.

[SW] E. Stein and G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, 1971.

[We] S. M. Webster, On the mapping problem for algebraic real hypersurfaces, Invention. Math. , 43 (1977), 53-68.

[We1] S. M. Webster, Pseudo-Hermitian structures on a real hypersurface, J. Diff. Geom. 13 (1978), 25-41.

[We2] S. M. Webster, Some birational invariants for algebraic real hypersurfaces, Duke Math. J. 45 (1978), 39-46.

[Z1] D. Zaitsev, Obstructions to embeddability into hyperquadrics and explicit examples, Math. Ann., 342 (2008), 695-726.

[Z2] D. Zaitsev, Algebraicity of local holomorphisms between real-algebraic submanifolds of complex spaces, Acta Math., 183 (1999), 273-305.

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