# NONCOMMUTATIVE RING THEORY NOTES

## Contents



## 1. Notational Conventions & Definitions

In these notes, $R$ always denotes a ring with 1, unless otherwise stated.

### 1.1. **Notation.**

subset, proper subset, ideal, right/left ideal, right module, bimodule

### 1.2. **Definitions.**

simple, semisimple, prime, semiprime, primitive, semiprimitive, artinian, noetherian, nil, nilpotent, regular, dedekind-finite, essential, uniform, goldie

## 2. Introduction

**Definition 2.1.** A *right $R$-module*, $M_R$, is an abelian group under addition together with a map $M_R \times R \to M_R$, written $(m, r) \mapsto mr$, such that the following hold, for all $m, n \in M_R$, and all $r, s \in R$:

(1) $(m+n)r = mr + nr$
(2) $m(r+s) = mr + ms$
(3) $m(rs) = (mr)s$
(4) $m \cdot 1 = m$

If the ring $R$ is understood, we usually drop the subscript and just write $M$ in place of $M_R$. A subgroup $N$ of $M_R$ is a *submodule* if $NR \subseteq N$. Moreover, if $N$ is a submodule of $M$, then we can form the factor module $M/N$ in the obvious way. As a set, we have $M/N = \{\, m+N \mid m \in M \,\}$, and the action of $R$ is given by $(m+N)r = mr + N$. We also have the concept of an *$R$-module homomorphism*, which is a map $\varphi : M_R \to M'_R$ such that $\varphi(m+n) = \varphi(m)+\varphi(n)$, and $\varphi(mr) = \varphi(m)r$, for all $m, n \in M_R, r \in R$. The image of a module under an $R$-module homomorphism is again an $R$-module, and all the usual isomorphism theorems are still true.

**Example 2.2.**

- A vector space is a module over a field.
- $R$ is naturally a right $R$-module, in which case the submodules of $R_R$ are exactly the right ideals of $R$.

**Exercise 2.3.** Take any ring $R$ and prove that the two-sided ideals of $M_n(R)$ are of the form $M_n(I)$, where $I$ is a two-sided ideal of $R$. Show that the analogous statement for one-sided ideals is *not* true.

**Corollary 2.4.** *$M_n(k) = \{n \times n$ matrices over the field $k\}$, is a simple ring.*

**Definition 2.5.** A module $M_R$ is *finitely generated* if there are elements $m_1, \dots, m_n \in M$ such that, given $m \in M$, there are elements $r_1, \dots, r_n \in R$ with $m = m_1r_1 + \cdots m_nr_n$. That is, $M_R = m_1R + \ldots + m_nR$.

**Example 2.6.** $R = k[x, y]$, the polynomial ring in two commuting variables over the field $k$. The ideal $I = xR + yR$ is a finitely generated $R$-module. (generated by $x$ and $y$).

**Remark 2.7.**

(1) If $M_R$ is finitely generated, and $\varphi$ is an $R$-module homomorphism, then $\varphi(M_R)$ is also finitely generated (by the images of the generators).
(2) A submodule of a finitely generated module needn't be finitely generated. For example, let $V$ be an infinite dimensional vector space over a field $K$. Let $R = \{\, (v, \alpha) \mid v \in V, \alpha \in K \,\}$ with operations

$$(v, \alpha) + (w, \beta) = (v + w, \alpha + \beta)$$

$$(v, \alpha)(w, \beta) = (\beta v + \alpha w, \alpha\beta).$$

$R$ is a ring with identity $(0,1)$, (and hence finitely generated by $(0,1)$ as a right $R$-module). However, $(V,0)$ is an ideal (and thus a right $R$-submodule) which is not finitely generated.

(3) $\begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{Q} \end{pmatrix} \subset M_2(\mathbb{R})$ is a ring, and $I = \begin{pmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{pmatrix}$ is a right ideal which is not finitely generated. (because $\mathbb{R}_{\mathbb{Q}}$ is not finitely generated).

**Definition 2.8.** We say a module $M_R$ is *noetherian* if every submodule of $M_R$ is finitely generated. Similarly, we say a ring $R$ is *right noetherian* if $R_R$ is a noetherian module.

**Theorem 2.9.** *The following are equivalent:*

(1) *$M_R$ is noetherian*

(2) *$M_R$ satisfies a.c.c. ( = ascending chain condition) on submodules: given any ascending chain*

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

*of submodules, there is an integer $r$ such that*

$$M_r = M_{r+1} = M_{r+2} = \cdots .$$

(3) *Any nonempty family of submodules of $M_R$ has a maximal element (with respect to inclusion).*

*Proof.* (1) $\Rightarrow$ (2) Suppose we have a chain of submodules

$$M_1 \subseteq M_2 \subseteq \cdots$$

Note that $\bigcup M_i$ is a submodule of $M$, which by hypothesis is finitely generated, by say $\{m_1, \ldots m_t\}$. Since this set is finite, there must be some $r$ for which all of the $m_i$ are in $M_r$. Thus $\bigcup M_i = \sum m_i R \subseteq M_r$. Hence $M_r = M_{r+1} = M_{r+2} = \cdots$, and the chain stabilizes.

(2) $\Rightarrow$ (3) Suppose $\mathcal{F}$ is a nonempty family of submodules of $M_R$. Choose $M_i \in \mathcal{F}$. If $M_i$ is maximal, we're done. Otherwise we can find $M_2 \in \mathcal{F}$ with $M_1 \subset M_2$. If $M_2$ is maximal, we're done. Otherwise we can find $M_3 \in \mathcal{F}$ with $M_2 \subset M_3$. Since we're assuming (2), this process must terminate, at which point we've arrived at a maximal element of $\mathcal{F}$.

(3) $\Rightarrow$ (1) Let $N$ be any submodule of $M_R$, and set

$$\mathcal{F} = \{\text{all finitely generated submodules of } N\}.$$

Let $N'$ be a maximal element of $\mathcal{F}$. If $N' = N$ we're done. Otherwise $N' \subsetneq N$, so we can find an element $n \in N \setminus N'$. Then $N' + nR$ is a finitely generated submodule of $N$ which properly contains $N'$, contradicting our choice of $N'$. $\square$

**Definition 2.10.** We say a module $M_R$ is *artinian* if it satisfies d.c.c (= descending chain condition) on submodules. That is, given any descending chain

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$$

of submodules, there is an integer $r$ such that

$$M_r = M_{r+1} = M_{r+2} = \cdots .$$

**Example 2.11.**

(1) Any finite dimensional vector space $V_k$ is an artinian $k$-module.
(2) $\mathbb{Z}_\mathbb{Z}$ is noetherian, but not artinian. For example the chain $2\mathbb{Z} \supsetneq 2^2\mathbb{Z} \supsetneq 2^3\mathbb{Z} \supsetneq \cdots$ doesn't terminate.
(3) If $R_R$ is a commutative integral domain with d.c.c., then $R$ is a field. (proof: Choose $0 \neq a \in R$ and consider the chain of ideals

$$aR \supseteq a^2R \supseteq a^3R \cdots$$

which must stabilize. So $a^nR = a^{n+1}R$ for some $n$. Then $a^n = a^{n+1}r$ for some $r \in R$, and since $R$ is a domain, we can cancel $a^n$ to get $1 = ar$, so $a$ is a unit.)
(4) $(\mathbb{Q}/\mathbb{Z})_\mathbb{Z}$ has a $\mathbb{Z}$-submodule

$$Z_{p^\infty} = \{\, g \in \mathbb{Q}/\mathbb{Z} \mid p^n g = 0 \text{ for some } n \in \mathbb{N} \,\},$$

which is artinian, but not noetherian. The $\mathbb{Z}$-submodules of $\mathbb{Z}_{p^\infty}$ are all of the form $\mathbb{Z}_{p^n} = \{\, g \in \mathbb{Q}/\mathbb{Z} \mid p^n g = 0 \,\}$, and we have a strictly increasing chain

$$\mathbb{Z}_p \subset \mathbb{Z}_{p^2} \subset \mathbb{Z}_{p^3} \subset \cdots$$

**Proposition 2.12.** *The following are equivalent for a right $R$-module $M$:*

(1) *$M_R$ has d.c.c. on submodules.*
(2) *Any nonempty family of submodules of $M_R$ has a minimal element (with respect to inclusion).*

*Proof.* Mimic the proof in the noetherian case. □

**Proposition 2.13.** *Any homomorphic image of a noetherian (resp. artinian) $R$-module is noetherian (resp. artinian).*

The converse of this result is a useful type of induction for noetherian (resp. artinian) modules.

**Proposition 2.14.** *Let $N$ be an $R$-submodule of $M$. If $N$ and $M/N$ are noetherian (resp. artinian) then $M$ is noetherian (resp. artinian).*

*Proof.* We will do the proof in the noetherian case. The artinian case is similar. Let

$$M_1 \subseteq M_2 \subseteq \cdots$$

be an ascending chain of submodules of $M$. Then

$$M_1 \cap N \subseteq M_2 \cap N \subseteq \cdots$$

is an ascending chain of submodules of $N$, and

$$(M_1 + N)/N \subseteq (M_2 + N)/N \subseteq \cdots$$

is an ascending chain of submodules of $M/N$. Both of these chains stabilize, so choose $r \in \mathbb{N}$ large enough so that

$$M_r \cap N = M_{r+1} \cap N = \cdots$$

and

$$(M_r + N)/N = (M_r + N)/N = \cdots .$$

Then

$$\begin{aligned} M_{r+1} &= M_{r+1} \cap (M_{r+1} + N) = M_{r+1} \cap (M_r + N) \\ &= M_r + (M_{r+1} \cap N) = M_r + (M_r \cap N) = M_r \end{aligned}$$

where the third equality follows from the modular law. Thus the chain stabilizes, so $M$ is noetherian. □

**Proposition 2.15.** *A finite direct sum of noetherian (resp. artinian) modules is again noetherian (resp. artinian)*

*Proof.* The proof is by induction on the number of modules. Suppose $M_1, \cdots, M_n$ are noetherian $R$-modules. We view $M_n$ as a submodule of $M_1 \oplus \ldots \oplus M_n$ in the usual way. Then $(M_1 \oplus \ldots \oplus M_n)/M_n \cong M_1 \oplus \ldots \oplus M_{n-1}$ is noetherian by induction. Now apply the previous proposition. □

Of course, the previous proposition is not true for infinite direct sums.

**Definition 2.16.** A ring $R$ is called right noetherian (resp. artinian) if $R_R$ is a noetherian (resp. artinian) $R$-module.

Note that any homomorphic image of a right noetherian (resp. artinian) ring is right noetherian (resp. artinian).

**Proposition 2.17.** *If $R$ is right noetherian (resp. artinian), then any finitely generated right $R$-module is noetherian (resp. artinian).*

*Proof.* Again, we will do the noetherian case only, as the artinian case is similar. Let $M$ be a finitely generated $R$-module, generated by $m_1, \cdots, m_n$. If $n = 1$, then $M = mR$. Set $I = \{ r \in R \mid mr = 0 \}$. Then $I$ is a right ideal of $R$, and we have an $R$-module isomorphism $M \cong R_R/I$. Since $R_R$ has a.c.c., so does $M$. Now suppose $M = m_1R + \ldots + m_nR$. Set $N = m_nR$, a submodule of $M$. Note that we have $M/N \cong \sum_{i=1}^{n-1} \overline{m_i}R$, where $\overline{m_i} = m_i + m_nR$. By induction, $M/N$ is noetherian, and by the first part, $N$ is noetherian, hence $M$ is noetherian as well. □

**Remark 2.18.** We have the following application: If $R_R$ satisfies a.c.c. (resp. d.c.c.) and $S \supseteq R$ is any ring such that $S_R$ is a finitely generated right $R$-module, then $S_S$ satisfies a.c.c. (resp. d.c.c.)

In particular, if $R$ satisfies a.c.c. (resp. d.c.c.) then so too does $M_n(R)$.

There is also the dual notion of left $R$-module, the definition of which should be clear. Everything we have said about right modules could be repeated for left modules. However some care should be taken not to confuse one's left and right. For example, we can have a module $M$ which is both a right and left $R$-module, but these module structures can be wildly different. Here is an amusing example.

**Example 2.19** (Small)**.** The ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is right noetherian but not left noetherian. To see this, note that $R$ is a finitely generated right $\begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Q} \end{pmatrix}$-module, (generated by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$). Moreover, $\begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Q} \end{pmatrix}$ is isomorphic to a direct sum of two noetherian rings, hence is noetherian. Thus we see that $R$ is right noetherian by the previous proposition. Next we need to show that $R$ is not left noetherian. To that end, note that $\begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$ is a left $R$-submodule of $R$ which is not finitely generated. (If it were, then $\mathbb{Q}$ would be a finitely generated $\mathbb{Z}$-module, which it's not. In fact, if $S$ is any commutative integral domain (not a field), then it's field of quotients is never a finitely generated $S$-module.)

Even if the ring $R$ is commutative, the right and left module actions of $R$ on itself needn't be the same. As a specific example, we have the following.

**Example 2.20.** Consider $M = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. $M$ is naturally a $\mathbb{C}$-module on both sides (the action is just multiplication), but these actions are not the same. $M$ is a 4-dimensional vector space over $\mathbb{R}$, with basis

$\{e_1 = 1 \otimes 1, e_2 = 1 \otimes i, e_3 = i \otimes 1, e_4 = i \otimes i\}$. But note that $ie_1 = e_3$, while $e_1 i = e_2$.

## 3. Rings with d.c.c.

**Definition 3.1.** An element $r \in R$ is *nilpotent* if $r^n = 0$ for some positive integer $n$. A right ideal $I \lhd_r R$ is *nil* if every element of $I$ is nilpotent, and $I$ is *nilpotent* if $I^n = 0$ for some $n \in \mathbb{N}$.

Note that nilpotent ideals are always nil, but not vice versa. For an example of a nil but not nilpotent ideal, consider a polynomial ring in countably many variables over a field $k[x_1, x_2, x_3, \ldots]$, and set $I = (x_1^2, x_2^3, x_3^4, \ldots)$. Let $R = k[x_1, x_2, x_3, \ldots]/I$, and write $\overline{x_i}$ for the image of $x_i$ in $R$. Then (check!) $(\overline{x_1}, \overline{x_2}, \ldots)$ is a nil ideal of $R$ which is not nilpotent.

Also, we have the following

**Proposition 3.2.**
(1) *If $I$ is a nilpotent right ideal of $R$, then $RI$ is a nilpotent two-sided ideal of $R$.*
(2) *If $I$ and $J$ are nilpotent right ideals of $R$, then $I + J$ is also a nilpotent right ideal.*
(3) *A ring $R$ has no nonzero nilpotent ideals iff $R$ has no nonzero nilpotent right ideals.*

**Definition 3.3.** A ring $R$ is called *semiprime* if $R$ has no nonzero nilpotent ideals.

**Remark 3.4** (Warning)**.** Semiprime rings with d.c.c. are sometimes referred to as *semisimple* rings with d.c.c. (though not in these notes) so beware. Really the situation is this: A ring $R$ is called *semiprime* if it has no nonzero nilpotent ideals, whereas $R$ is *semisimple* if $R$ is (isomorphic to) a finite direct sum of simple rings. (simple = no nontrivial ideals). The confusion arises because if $R$ has d.c.c. on right or left ideals, then these two concepts are the same, as we will see.

**Proposition 3.5.** *Suppose $R$ is a ring with no nonzero nilpotent right ideals. If $I \lhd_r R$ is a minimal right ideal of $R$, then $I = eR$ for some idempotent $e \in R$. (recall that $e \in R$ is called* idempotent *if $e^2 = e$.)*

*Proof.* Since $I^2 \neq 0$, there is some $x \in I$ such that $xI \neq 0$. Since $xI$ is a right ideal of $R$ which is contained in $I$, we must have $xI = I$, since $I$ is minimal. Since $x \in I = xI$, we can find some $e \in I$ such that $xe = x$, which implies that $xe^2 = xe$, or $x(e^2 - e) = 0$. Now consider the set $\mathrm{r.ann}_I(x) = \{\, a \in I \mid xa = 0 \,\}$. This is a right ideal of $R$ contained in $I$, and since $xI \neq (0)$, we have $\mathrm{r.ann}_I(x) = 0$, again by minimality of $I$. Now, since $x(e^2 - e) = 0$, we have $e^2 - e \in \mathrm{r.ann}_I(x) = 0$, thus $e^2 = e$.

Lastly, since $e \in I$, $eR$ is a right ideal contained in $I$, and is nonzero since $0 \neq e \in eR$. Thus $eR = I$ by minimality of $I$ once again. $\square$

**Proposition 3.6.** *Suppose $R$ has no nonzero nilpotent ideals and let $e \in R$ be idempotent. Then $eR$ is a minimal right ideal of $R$ iff $eRe$ is a division ring (with $e$ as identity element).*

*Proof.* ($\Leftarrow$) Suppose $eRe$ is a division ring (it is clear that the identity element must then be $e$) and $I \lhd_r R$ is a nonzero right ideal of $R$ contained in $eR$. Then for any $er \in I$, we have $e(er) = er$, so $eI = I$. Next, since $I \neq 0$, we have $(eI)^2 \neq 0$ because $R$ has no nonzero nilpotent right ideals. This then implies that $eIe \neq 0$. Hence there is some $a \in I$ with $eae \neq 0$. Since $eRe$ is a division ring, we can find $ese \in eRe$ such that $(eae)(ese) = e$. But then $e = (eae)(ese)$ is an element of $I$, and so $eR \subseteq I$. As we assumed that $I \subseteq eR$, we have $eR = I$, and so $eR$ is minimal.

($\Rightarrow$) Suppose $eR$ is a minimal right ideal of $R$. Choose $r \in R$ with $ere \neq 0$, which we can find since $R$ has no nonzero nilpotent right ideals. Then $0 \neq ereR \subseteq eR$, so $ereR = eR$ by minimality of $eR$. Since $e^2 = e \in R$, we can find $s \in R$ such that $eres = e$. Then $(ere)(ese) = e$, so $ere$ has an inverse in $eRe$, and $eRe$ is a division ring. $\square$

**Remark 3.7.** Note that the hypothesis that $R$ have no nonzero nilpotent ideals is necessary. For let $k$ be a field, and consider the ring $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$. We have $e = e_{11} \in R$ is idempotent, and $eR = \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}$ is a right ideal of $R$ which is not minimal since it contains the right ideal $\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$. But $eRe \cong k$ is a field, so also a division ring.

In a semiprime ring which is also right artinian, every right ideal is generated by an idempotent. This is the next

**Proposition 3.8.** *Let $R$ be a semiprime right artinian ring. If $I \lhd_r R$ is any right ideal of $R$, then $I = eR$ for some idempotent $e$.*

*Proof.* Let $I$ be a right ideal of $R$. If $I$ is minimal then $I = eR$ for some idempotent $e$ and we're done. Otherwise, since $R$ is artinian, $I$ contains a minimal right ideal which must be of the form $eR$ for some idempotent $e$. Let $e$ be such that $\mathrm{r.ann}_I(e) = \{ r \in I \mid er = 0 \}$ is minimal (which we can do since $R$ is artinian). If $\mathrm{r.ann}_I(e) = 0$, then (since $e^2 = e$) for any $x \in I$, we have $0 = e(ex - x)$. So $ex - x \in \mathrm{r.ann}_I(e) = 0$, whence $ex = x$, and thus $I \subseteq eR$, so $I = eR$.

If $\mathrm{r.ann}_I(e) \neq 0$, then we can find $0 \neq e' \in \mathrm{r.ann}_I(e)$ with $(e')^2 = e'$. Set $e^* = e + e' - e'e$. Then $(e^*)^2 = e^*$, and $ee^* = e \neq 0$, so $e^* \neq 0$. Moreover, since $e^*r = 0 \implies er = ee^*r = 0$, we have $\mathrm{r.ann}_I(e^*) \subseteq \mathrm{r.ann}_I(e)$ so $\mathrm{r.ann}_I(e^*) = \mathrm{r.ann}_I(e)$ be our choice of $e$. Now note that $e' = e^*e' \neq 0$, whereas $ee' = 0$, so $\mathrm{r.ann}_I(e^*) \subset \mathrm{r.ann}_I(e)$ which contradicts our choice of $e$. We conclude that $\mathrm{r.ann}_I(e) = 0$ and so $I = eR$ as desired. □

**Proposition 3.9.** *Let $R$ be semiprime right artinian. If $I$ is an ideal of $R$ with $I = eR$ for some idempotent $e$, then $I = Re$.*

*Proof.* Pick $x \in I$. Then $(xe - x)e = 0$, so $(xe - x)eR = 0$. If $xe - x \neq 0$, then $xe - x \in \mathrm{l.ann}_R(I) = \{ r \in R \mid rI = 0 \}$ is a nonzero ideal contained in $I$. We claim that $\mathrm{l.ann}_R(I)^2 = 0$. Since $e$ is a right identity element for $I$, we have $\mathrm{l.ann}_R(I)e = \mathrm{l.ann}_R(I)$, whence $\mathrm{l.ann}_R(I)^2 = (\mathrm{l.ann}_R(I)e)^2 = 0$. Since $R$ has no nonzero nilpotent ideals, we have $\mathrm{l.ann}_R(I) = 0$, so $xe = x$ and $I = Re$. □

**Corollary 3.10.** *Let $R$ be a semiprime right artinian ring, and $I = eR = Re$ an ideal of $R$. Then $e$ is in the center of $R$.*

*Proof.* By the previous proposition, we see that $e$ acts as the identity element on $I$. So pick $x \in R$ and note that since $ex$ and $xe$ are in $I$, we have $ex = (ex)e = e(xe) = xe$. □

**Exercise 3.11.** If $R$ has no nonzero nilpotent *elements*, then every idempotent is in the center of $R$.

Summarizing, we see that if $R$ is semiprime right artinian, then every right (or left) ideal is generated by an idempotent, and every two-sided ideal is generated by a central idempotent.

**Definition 3.12.** A ring $R$ is *simple* if $R$ has no nontrivial proper two-sided ideals. (ie the only two-sided ideals are 0 and $R$).

**Example 3.13.**

- Any field, or more generally any division ring, is simple.
- $M_n(D)$ is simple, for $D$ any division ring (by a previous exercise).
- More generally, $M_n(R)$ is simple iff $R$ is simple.

Note that if $D$ is a division ring, then $M_n(D)$ is a finite dimensional vector space over $D$, so is artinian. Thus we have examples of simple artinian rings. The first part of the Artin-Wedderburn theorem says that these are *all* the simple artinian rings. Moreover, Artin-Wedderburn characterizes all semisimple artinian rings as finite direct sums of matrix rings over division rings. This is what we now aim for.

**Theorem 3.14** (Schur's Lemma)**.** *If $M$ is a simple right $R$-module ( = has no nonzero proper submodules), then* $\operatorname{End}_R(M)$ *is a division ring.*

*Proof.* Just note that kernels and images of $R$-module homomorphisms are submodules. More specifically, suppose $\varphi \in \operatorname{End}_R(M)$. If $\varphi \neq 0$, then $\varphi(M)$ is a nonzero submodule of $M$, hence $\varphi(M) = M$ since $M$ is simple. Similarly, $\ker \varphi \neq M$ since $\varphi \neq 0$, so $\ker \varphi = 0$. Thus $\varphi$ is an isomorphism, and $\operatorname{End}_R(M)$ is a division ring. $\square$

**Proposition 3.15.** *Let $R$ be a simple artinian ring and let $eR$ be a minimal right ideal (which is thus a simple $R$-module). Then $eR$ is a finite dimensional (left) vector space over the division ring $eRe$.*

*Proof.* All that remains to be shown is that $eR$ is finite dimensional over $eRe$. Note that $ReR$ is a nonzero ideal of $R$, hence $ReR = R$ by simplicity of $R$. We can then write $1 = \sum r_i e s_i$ for some finite collection of elements $r_i, s_i \in R$. Then $er = er1 = \sum (err_i e)s_i$, so $eR \subseteq \sum (eRe)s_i$, which shows that $eR$ is finite dimensional over $eRe$. $\square$

**Proposition 3.16.** *Let $R$ be a simple artinian ring, then all simple $R$-modules are isomorphic.*

*Proof.* Let $eR$ be a minimal right ideal, and $M = M_R$ a simple right $R$-module. We will show that $M \cong eR$ as $R$-modules. First, $\operatorname{ann}_R(M) = \{ r \in R \mid Mr = 0 \}$ is a proper two-sided ideal of $R$, so $\operatorname{ann}_R(M) = 0$ since $R$ is simple. Thus $MeR = M$ since $MeR$ is a nonzero submodule of $M$, and $M$ is simple. We can then conclude that there is $m \in M$ such that $meR \neq 0$, whence $meR = M$, again by simplicity of $M$. So we get an $R$-module homomorphism $\varphi : eR \to M$ given by $\varphi(er) = mer$, and of course $\varphi$ is an isomorphism since $eR$ and $M$ are simple modules. $\square$

**Theorem 3.17.** *Let $R$ be a semiprime artinian ring, then every $R$-module is a finite direct sum of simple submodules.*

WE SHOULD REDO THE NEXT TWO THEOREMS TO PERTAIN TO RIGHT IDEALS

AND MAKE CLEAR WHICH SIDE OF $eR$ THINGS ARE ACTING ON

**Proposition 3.18.** *Let $e \in R$ be idempotent, then* $\operatorname{End}_R(eR) \cong eRe$.

*Proof.* Consider the map $f \mapsto f(e)e$ from $\operatorname{End}_R(eR)$ to $eRe$. We claim that this is an $R$-module isomorphism. Choose $f \in \operatorname{End}_R(eR)$, and suppose $f(e) = et$, then $f(er) = f(e)r = f(e)er = (ete)er$, so $f$ acts on $eR$ by left multiplication by $f(e)e = ete \in eRe$, yielding the isomorphism. $\square$

**Theorem 3.19** (Rieffel)**.** *Let $R$ be a simple ring, and $0 \neq A$ a left ideal of $R$. Set $R' = \mathrm{End}_R(A)$ (viewed as a ring of* right *operators on $A$), then $R \cong \mathrm{End}_{R'}(A)$ as rings.*

*Proof.* To clarify, since $R$ acts on the left of $A$, we are thinking of $\mathrm{End}_R(A)$ as a ring of *right* operators on $A$, in contrast to $\mathrm{End}_{R'}(A)$, which we thing of as acting on the *left* of $A$. That is, $A$ is naturally an $(R, R')$-bimodule, and we show that the ring of $R'$-module endomorphisms consists of left multiplication by elements of $R$.

Let $\lambda : R \to \mathrm{End}_{R'}(A)$ denote the ring homomorphism sending $r \in R$ to left multiplication by $r$. $\ker \lambda$ is a two-sided ideal of $R$, and since $R$ is simple, $\lambda$ is injective. Now, $0 \neq AR \lhd R$, and again since $R$ is simple, we must have $AR = R$, and thus $\lambda(A)\lambda(R) = \lambda(R)$. Now, for any $x, y \in A$, and $f \in \mathrm{End}_{R'}(A)$, we have $f(xy) = f(x)y$, because right multiplication by $y$ is an $R$-endomorphism of $A$. Hence $\lambda(A)$ is a left ideal of $\mathrm{End}_{R'}(A)$, and so

$$\mathrm{End}_{R'}(A) = \mathrm{End}_{R'}(A)\lambda(R) = \mathrm{End}_{R'}(A)\lambda(A)\lambda(R) = \lambda(A)\lambda(R) = \lambda(R)$$

which shows that $\lambda$ is surjective as well. □

**Remark 3.20.** We have the following "duality" between $R$ and $R'$ above: $R \cong \mathrm{End}_{R'}(A)$ and $R' \cong \mathrm{End}_R(A)$. For this reason, the above theorem is sometimes referred to as the "Double Centralizer Property".

**Corollary 3.21.** *If $R$ is simple right artinian, then we have a ring isomorphism $R \cong M_n(D)$, where $D$ is a division ring.*

*Proof.* Let $eR$ be a minimal right ideal of $R$. Then by Schur's Lemma, $D = eRe \cong \mathrm{End}_R(eR)$ is a division ring (acting on the *left* of $eR$), and we have a ring isomorphism $R \cong \mathrm{End}_D(eR)$ (acting on the right of $eR$). Since $\dim_D(eR) = n < \infty$, we see that $R \cong M_n(D)$, as desired. □

**Remark 3.22.** The integer $n$ (and the division ring $D$) are both uniquely determined by $R$. Moreover, since a matrix ring over a division ring is both right and left artinian, we see that a simple right artinian ring is also left artinian.

**Theorem 3.23.** *Let $R$ be a ring with nonzero ideals $B_1, \ldots, B_r$ and $C_1, \ldots, C_s$ with*

$$R = B_1 \oplus \cdots \oplus B_r = C_1 \oplus \cdots \oplus C_s$$

*and such that each $B_i$ as well as each $C_i$ is not a direct sum of two nonzero subideals. Then $r = s$ and after a permutation of indices, $B_i = C_i$ for all $i$.*

*Proof.* Viewing the $B_i$'s as rings, $R \cong B_1 \oplus \cdots \oplus B_r$. Under this isomorphism, the ideal $C_1$ corresponds to an ideal $I_1 \oplus \cdots \oplus I_r$, where each $I_i \lhd B_i$. Since $C_1$ is not a direct sum of subideals, all but one of these $I_i$ is zero. We may thus assume that $C_1 = I_1$, so $C_1 \subseteq B_1$. Similarly, we get $B_1 \subseteq C_i$ for some $i$. But then $C_1 \subseteq C_i$ implies that $i = 1$. Repeating this argument for the other $C_i$'s finishes the proof. □

**Theorem 3.24** (Wedderburn-Artin)**.** *Let $R$ be a semiprime right artinian ring. Then*

$$R \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_r}(D_r)$$

*for suitable division rings $D_1, \ldots, D_r$ and positive integers $n_1, \ldots, n_r$. Moreover, the integer $r$, as well as the pairs $(n_i, D_i)$ are uniquely determined (up to permutation).*

3.1. **Finite Dimensional Algebras.** We now discuss Wedderburn's Theorem in the context of finite dimensional algebras over fields. As we will see, in this case things behave even nicer than in the general case of rings. We start with a definition.

**Definition 3.25.** Let $K$ be a field. An *algebra over $K$*, or a *$K$-algebra*, is a ring $A$ which is also a vector space over $K$. The ring structure and the vector space structure are related by demanding that

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \text{ for all } \alpha \in K, x, y \in A.$$

Note that if $A$ has an identity element, then the condition just says that $A$ contains $K$ in it's center. We say that $A$ is a finite dimensional algebra if $A$ is finite dimensional as a vector space over $K$.

For the remainder of this section, $A$ is a finite dimensional $K$ algebra. We will be particularly interested in finite dimensional semisimple $K$-algebras. First, if $A$ is simple, then we know that $A \cong M_n(D)$ for some division ring $D$. Thus $A$ must have a 1, and moreover we view $K$ as contained in $A$ as the set of scalar matrices $\{\, \alpha I \mid \alpha \in K \,\}$. Also, we have $D$ as a subring of $A$ as the set of scalar matrices $\{\, dI \mid d \in D \,\}$. Then we see that $K \subseteq D \subset A = M_n(D)$. Since $A$ is finite dimensional over $K$, so too is $D$. It is an easy exercise to see that if $K$ is algebraically closed, then in fact $D = K$, from which it follows that in case $K$ is algebraically closed, $A \cong M_n(K)$.

Next suppose that $K$ is algebraically closed, and that $A$ is semisimple and finite dimensional over $K$. It follows just as before that $A \cong \bigoplus M_{n_i}(K)$ is a direct sum of full matrix algebras over $K$. This is basically the original form of Wedderburn's theorem (though he was really only interested in the case $K = \mathbb{C}$). Artin later modified Wedderburn's result to apply to semisimple rings with d.c.c.

We wish to prove another theorem of Wedderburn, but first we need a small digression to discuss the process of "extending the base field". Suppose $A$ is a $K$-algebra, and $F$ is a field extension of $K$. We can then form the tensor product $A \otimes_K F$, which is spanned, as a vector space over $K$, by all simple tensors $a \otimes \alpha$, where $a \in A$ and $\alpha \in F$. (remark: elements of $A \otimes_K F$ are (finite) *sums* of simple tensors!) The tensor product is bilinear, and in addition satisfies

$$k(a \otimes \alpha) = (ka) \otimes \alpha = a \otimes (k\alpha) = (a \otimes \alpha)k, \text{ for all } a \in A, \alpha \in F, k \in K.$$

We make $A \otimes_K F$ into a ring by defining multiplication between simple tensors via

$$(a \otimes \alpha)(b \otimes \beta) = ab \otimes \alpha\beta$$

and extending this product linearly. Note that since $K$ is central, we have that $A \otimes_K F$ is a $K$-algebra. But moreover, we can view $F$ as contained in $A \otimes_K F$ as the set of all elements of the form $1 \otimes \alpha$, and in this way (since $1 \otimes \alpha$ is central for all $\alpha \in F$), we see that $A \otimes_K F$ is also an $F$-algebra. Moreover, we see that the dimension of $A \otimes_K F$ over $F$ is the same as the dimension of $A$ over $K$:

$$\dim_F A \otimes_K F = \dim_K(A).$$

With this in hand, we can now prove the following pleasant result

**Theorem 3.26** (Wedderburn)**.** *Let $A$ be a finite dimensional $K$-algebra in which every element of $A$ is a sum of nilpotent elements. Then $A$ is nilpotent.*

*Proof.* If $K$ is not algebraically closed (ie $K \neq \overline{K}$, consider $\tilde{K} := A \otimes_K \overline{K}$. Then $\dim_{\tilde{K}}(A \otimes_K \overline{K}) = \dim_K(A)$, and every element of $A \otimes_K \overline{K}$ is again a sum of nilpotents. Thus without loss, we may assume that $K$ is algebraically closed. Let $N$ denote the radical of $A$, which is nilpotent since $A$ is artinian. If $N = A$, then we're done. Otherwise, consider $\overline{A} = A/N$, and note that every element of $\overline{A}$ is also a sum of nilpotents (because this condition clearly passes to quotient algebras). By Wedderburn's theorem, $\overline{A} \cong \bigoplus M_{n_i}(K)$. Since each $M_{n_i}(K)$ is clearly an ideal of $\overline{A}$, we see that $\overline{A}$ has, say, $M_{n_1}(K)$ as a homomorphic image. Thus we conclude that every element of $M_{n_1}(K)$ is a sum of nilpotent elements. But this is absurd, for a nilpotent matrix must have trace zero, whereas a sum of matrices of trace zero needn't have trace zero. $\square$

## 4. Nil and Nilpotent Ideals

Recall that a ring $R$ is called *semiprime* if it has no nonzero nilpotent ideals. In this section we are interested in semiprime (right) noetherian rings, so we begin with some examples of such.

**Example 4.1.** The following are semiprime (right) noetherian.

- Any domain.
- $\mathbb{Z}, M_n(\mathbb{Z}), F[x]$
- $F[x,y]/(xy)$ has no nilpotent elements, so is semiprime. Note that this ring is not a domain.

**Theorem 4.2** (Hopkins-Levitzki)**.** *If $R$ has d.c.c. on right ideals, then any nil right or left ideal is nilpotent.*

*Proof.* Let $I \lhd_r R$ be a nil right ideal of $R$, and consider the descending chain of right ideals
$$I \supseteq I^2 \supseteq I^3 \cdots .$$
Since $R$ is right artinian, this chain must terminate, so we can find $n \in \mathbb{N}$ such that $I^n = I^{n+1} = I^{n+2} = \cdots$. If $I^n = 0$, we're done. Otherwise, among all right ideals $J$ such that $JI^n \neq 0$, choose a minimal one, say $J_0$. Fix an element $a \in J_0$, so that we have
$$(aI^n)I^n = aI^n \neq 0$$
Thus $aI^n$ is a right ideal contained in $J_0$, so by minimality of $J_0$, we have $aI^n = J_0$. We can then find $x \in I^n$ with $ax = a$. But this says $(1-x)a = 0$, and since $x$ is nilpotent, $1-x$ is invertible, and thus $a = 0$, a contradiction. ☐

**Theorem 4.3.** *If $Rx$ is a nil left ideal, then $xR$ is a nil right ideal.*

*Proof.* Choose $xr \in xR$. Then $(xr)(xr)(xr)\cdots = x(rx)(rx)r\cdots$, which is eventually 0 since $rx$ is nilpotent. ☐

**Theorem 4.4** (Hopkins)**.** *If $R$ is a right (resp. left) artinian ring (with 1), then $R$ is right (resp. left) noetherian.*

*Proof.* We need to show that every right ideal of $R$ is finitely generated. Let
$$\mathcal{F} = \{\, I \lhd_r R \mid I \text{ is not finitely generated} \,\}$$
and choose $I \in \mathcal{F}$ which is minimal. Let $N$ denote the radical of $R$. If $IN = 0$, then $I$ is a right $R/N$-module under the action $a(r+N) = ar$. (This is well defined precisely because $N$ annihilates $I$.) Moreover, the right $R$-module structure of $I$ is exactly the same as the right $R/N$-module structure of $I$, because all elements of $N$ act as 0. Now, since $R/N$ is semiprime right artinian, as a right $R/N$-module, $I$ is a direct

sum of simple right $R/N$-modules. But since $I$ satisfies d.c.c. on $R/N$-submodules, this sum must be finite. Since a simple $R/N$-module is generated by a single element, we see that $I$ is finitely generated as a right $R/N$-module, hence also as right $R$-module.

Next, suppose $IN \neq 0$. Then $IN \subseteq I$, and this must be a proper containment. For if $IN = I$, then $I = IN^k$ for all $k \in \mathbb{N}$, and since $N$ is nilpotent, $I$ would be 0, a contradiction. Hence $IN \subset I$. Consider the factor module $I/IN$, a right $R$-module. We have $(I/IN)N = 0$, so by the first part, $I/IN$ is finitely generated as a right $R/N$-module, and thus also as a right $R$-module. Since $IN$ is strictly contained in $I$, $IN \notin \mathcal{F}$, ie. $IN$ is a finitely generated $R$-module. Thus $I$ is a finitely generated extension of a finitely generated right $R$-module, so is finitely generated as well. □

Note that the converse is false, for $\mathbb{Z}$ is a ring which is noetherian but not artinian. Moreover, the theorem is false for modules, as $\mathbb{Z}_{p^\infty}$ is an artinian but not noetherian $\mathbb{Z}$-module. Also, the assumption that $R$ has an identity is necessary, as the following example shows.

**Example 4.5.** Consider the abelian group $\mathbb{Z}_{p^\infty}$. Defining multiplication between any two elements to be zero makes $\mathbb{Z}_{p^\infty}$ into a ring (without 1) which is artinian, but not noetherian.

**Theorem 4.6** (Hopkins)**.** *Let $R$ satisfy d.c.c. on right ideals, and set $N = \sum\{$nilpotent ideals$\}$. Then $N$ is nilpotent.*

*Proof.* Let $\widetilde{N}$ be a maximal nilpotent ideal of $R$, which we can find since $R$ is noetherian. If $U$ is any nilpotent ideal of $R$, then $\widetilde{N} + U$ is also nilpotent. Since $\widetilde{N} + U \supseteq \widetilde{N}$, we have $\widetilde{N} + U = \widetilde{N}$ by our choice of $\widetilde{N}$. Thus $U \subseteq \widetilde{N}$, which proves the claim. □

We have shown that if $R$ is right artinian, then there is a maximal nilpotent ideal $N$ of $R$ which, of course, contains all nil right or left ideals of $R$. This ideal $N$ is called the *(Jacobson) radical* of $R$.

**Proposition 4.7.** *$R/N$ has no nonzero nilpotent ideals.*

*Proof.* Nonzero ideals of $R/N$ are of the form $I/N$ where $N \subset I \lhd R$, and $(I/N)^k = N$ iff $I^k \subseteq N$. Since $N$ is nilpotent, we have $(I^k)^n = 0$ where $n$ is the index of nilpotence of $N$. Hence $I \subseteq N$, a contradiction. □

**Theorem 4.8** (Hopkins)**.** *If $R$ satisfies a.c.c. on right ideals, then $\sum\{$nilpotent ideals$\}$ is nilpotent.*

*Proof.* □

**Theorem 4.9.** *If $R$ is semiprime noetherian, then $R$ has no nonzero nil one-sided ideals.*

*Proof.* Suppose $I$ is a nonzero nil right ideal. Choose $a \in I$ such that $\mathrm{r.ann}(a) = \{ x \in R \mid ax = 0 \}$ is maximal (which is guaranteed to exist since $R$ is right noetherian). Next we claim that $aRa \neq 0$, for if $aRa = 0$, then $aR$ is a right ideal which squares to 0, hence is 0, which implies that $a = 0$. Now, since $aRa \neq 0$, $\exists b \in R$ such that $aba \neq 0$. But $\mathrm{r.ann}(a) \subseteq \mathrm{r.ann}(aba)$, so by our choice of $a$, we must have $\mathrm{r.ann}(a) = \mathrm{r.ann}(aba)$. Thus $ababa \neq 0$, since otherwise $aba = 0$. Repeating this argument, we get $(ab)^n a \neq 0$ for all $n \in \mathbb{N}$, hence $ab$ is a nonnilpotent element of $I$, so $I$ is not nil.

Note that if we had instead started with a left ideal $I$, then we would have $ba$ is a nonnilpotent element of $I$, since $(ab)^n a = a(ba)^n \neq 0$ for all $n \in \mathbb{N}$. □

**Theorem 4.10** (Levitzki)**.** *Let $R$ be right noetherian. If $I$ is a nil one-sided ideal, then $I$ is nilpotent.*

*Proof.* Let $N$ be a maximal nilpotent ideal, so $R/N$ is semiprime, and let $I$ be a nil right ideal. If $I \subseteq N$, then we're done. Otherwise, Consider $(I + N)/N$, a nil right ideal of $R/N$. By the previous result, $R/N$ has no nonzero nil right ideals, so $I + N \subseteq N \implies I \subseteq N$. □

**Definition 4.11.** A ring $R$ is called *Dedekind-finite* if $xy = 1 \implies yx = 1$.

**Example 4.12.** We construct a ring which is not Dedekind-finite. Let $V$ be a countable dimensional vector space over a field $F$, with basis $\{v_1, v_2, v_3, \cdots\}$, and set $R = \mathrm{End}_F(V)$. Define two "shift operators" in $R$ as follows: $T : v_i \mapsto v_{i+1}$, and $S : v_i \mapsto v_{i-1}$ for $i \geq 2$ and $S : v_1 \mapsto 0$. Then we have $ST = 1$, but $TS \neq 1$.

**Proposition 4.13.** *If $R$ is right noetherian, then $R$ is Dedekind-finite.*

*Proof.* Let $ab = 1$, and suppose that $ba \neq 1$. Now, we can form a chain of right ideals, $\mathrm{r.ann}(a) \subseteq \mathrm{r.ann}(a^2) \subseteq \mathrm{r.ann}(a^3) \subseteq \cdots$, and since $R$ is right noetherian, this chain must terminate. Next, $ab = 1 \implies a^n b^n = 1$ for any $n \in \mathbb{N}$, so by replaicing $a$ and $b$ by suitable powers, we may assume that $\mathrm{r.ann}(a) = \mathrm{r.ann}(a^2)$. Next, we have $0 = a(1 - ba)$, so $1 - ba \in \mathrm{r.ann}(a)$. We also have $a^2(1 - b^2a^2) = 0$, so $a(1 - b^2a^2) = 0$. But then $0 = a - ba^2 = (1 - ba)a$, and multiplying on the right by $b$ yields $1 = ba$, a contradiction. □

Let $R$ be a ring with elements $a, b$ such that $ab = 1$ but $ba \neq 1$. For all $i, j \in \mathbb{N}$, set

$$e_{ij} = (b^{i-1}a^{j-1} - b^i a^j)$$

Then (check!) the set $\{e_{ij}\}$ forms an infinite set of matrix units, in the sense that they satisfy $e_{ij}e_{kl} = \delta_{jk}e_{il}$. This shows that any ring which is not Dedekind-finite is neither artinian nor noetherian (on either side).

## 5. Quotient Rings

Given a ring $R$, whose elements aren't necessarily units, we're interested in when we can "invert" some subset of the elements of $R$. That is, we'd like to construct an overring of $R$ where some collection of elements of $R$ become invertible. This is a process called localization, and though it is easy to do for, say commutative integral domains, in the noncommutative case things get a bit trickier.

**Definition 5.1.** An element $r \in R$ is called *regular* if $r$ is not a zero divisor.

Let $R$ be a ring, and $Q \supseteq R$ an overring with the property that given $q \in Q$, we can write $q = ac^{-1}$ for some $a, c \in R$. Here $c^{-1} \in Q$ is a two-sided inverse of $c$. Let $\mathcal{S} = \{ r \in R \mid r \text{ is invertible in } Q \}$. Then $\mathcal{S}$ is multiplicatively closed, and consists of regular elements.

**Definition 5.2** (Ore Condition)**.** A subset $\mathcal{S}$ of a ring $R$ satisfies the *right Ore condition* if given $a \in R$, $s \in \mathcal{S}$, there are elements $b \in R$, $s_1 \in \mathcal{S}$, such that $as_1 = sb$. Equivalently, $\mathcal{S}$ must satisfy $a\mathcal{S} \cap sR \neq \emptyset$ for all $a \in R$, $s \in \mathcal{S}$.

**Remark 5.3.** If $R$ is a domain, and $\mathcal{S} = R \setminus 0$, then the right Ore condition assumes the following simple form: For all nonzero $a, s \in R$, we must have $aR \cap sR \neq 0$. Such rings are called *right Ore domains*.

Note that in the situation above, where $Q \supseteq R$ is an overring whose elements are of the form $ac^{-1}$, the set $\mathcal{S} = \{ r \in R \mid r \text{ is invertible in } Q \}$ clearly satisfies the right Ore condition.

If $\mathcal{S}$ is any multiplicatively closed subset of $R$ which satisfies the right Ore condition, then we can localize with respect to $\mathcal{S}$. That is, there exists an overring $R_{\mathcal{S}} \supseteq R$ with the following properties

(1) If $s \in \mathcal{S}$, then $s$ is invertible in $R_{\mathcal{S}}$
(2) Given $x \in R_{\mathcal{S}}$, we can write $x = as^{-1}$ for some $a \in R$, and $s \in S$.

We're particularly interested in the case $\mathcal{S} = \{\text{all regular elements of } R\}$. In the favorable situation, where $\mathcal{S}$ satisfies the right Ore condition, we call $R_{\mathcal{S}}$ the *classical quotient ring of* $R$, sometimes denoted $Q(R)$, and call $R$ an *order* in $R_{\mathcal{S}}$. As an easy example, if $R$ is any commutative integral domain, then $\mathcal{S} = \{\text{all regular elements of } R\} = R \setminus 0$, and $\mathcal{S}$

is obviously right Ore, so $R$ has a classical quotient ring $Q(R)$. This is precisely how one forms, eg. $\mathbb{Q}$ from $\mathbb{Z}$.

As another example, consider $M_2(\mathbb{Z})$. Any matrix $x \in M_2(\mathbb{Q})$ can be written as $x = AC^{-1}$, where $A \in M_2(\mathbb{Z})$, and $C = cI$ is a scalar matrix in $M_2(\mathbb{Z})$. (By finding a common denominator $c$ for the entries of $x$.) Thus $Q(M_2(\mathbb{Z})) = M_2(\mathbb{Q})$, and $M_2(\mathbb{Z})$ is an order in $M_2(\mathbb{Q})$.

Here's one more. Set $R = F[x,y]/(xy)$, where $F$ is a field. Then $R$ is semiprime noetherian, and $Q(R) \cong F(x) \oplus F(y)$. (check!)

We now compute $R_{\mathcal{S}}$ in the case where $R$ is artinian. So let $a \in R$ be regular. We have a descending chain of right ideals $aR \supseteq a^2R \supseteq \cdots$, which must terminate. So there is some minimal $n \in N$ such that $a^nR = a^{n+1}R$. It then follows that $a^n = a^{n+1}r$ for some $r \in R$, so $a^n(1 - ar) = 0$. Since $a$ is regular, we can cancel to get $ar = 1$, so $a$ is right invertible. Since $R$ is also left artinian, we can repeat this argument on the left to see that $a$ is also left invertible. We have thus shown that if $R$ is artinian, then every regular element is invertible, so $R$ is it's own classical ring of quotients.

Of course, not all rings will have a classical quotient ring. For example, let $F\langle x,y\rangle$ denote the free algebra over a field $F$ in the (noncommuting) variables $x$ and $y$. Then $F\langle x,y\rangle$ is a domain, but $xF\langle x,y\rangle \cap yf\langle x,y\rangle = 0$, so $F\langle x,y\rangle$ is not right Ore.

Of course, the free algebra above isn't noetherian, and one might suspect that right noetherian integral domains are Ore. This is the content of the next

**Theorem 5.4** (Goldie)**.** *A right noetherian integral domain is right Ore.*

*Proof.* Let $R$ be a right noetherian integral domain, and choose $0 \neq a, b \in R$. If $aR \cap bR \neq 0$ we're done, otherwise $ab \in aR$, and since $aR \cap bR = 0$, $ab \notin bR$. Thus the right ideal $bR + abR$ properly contains $bR$. Similarly, the right ideal $bR + abR + a^2bR$ properly contains $bR + abR$. Iterating this process produces a chain of right ideals

$$bR \subset bR + abR \subset bR + abR + a^2bR \subset bR + abR + a^2bR + a^3bR \subset \cdots.$$

Since $R$ is right noetherian, this chain must terminate, so there is some minimal $n \in \mathbb{N}$ such that

$$a^nbR \subseteq bR + abR + \ldots + a^{n-1}bR.$$

which yields

$$a^n b = bx_0 + abx_1 + \ldots + a^{n-1}bx_{n-1}, \text{ for some } x_i \in R.$$
$$\implies -bx_0 = a(bx_1 + \ldots + a^{n-2}bx_{n-1} - a^{n-1}b)$$

and by minimality of $n$, neither side is zero. But this is a contradiction, because it shows that $bR \cap aR \neq 0$ □

The noetherian hypothesis is actually not entirely necessary, all that is really needed is that the domain $R$ does not contain an infinite direct sum of right ideals, and this is what Goldie originally noticed.

In some sense the free algebra example above is essentially the only sort of domain without a classical ring of quotients. More precisely, we have the following

**Theorem 5.5.** *If $R$ is an integral domain (containing a central field $F$) which is not right Ore, then $R$ contains a free $F$-algebra on two variables.*

*Proof.* Since $R$ is not right Ore, there are elements $x, y \in R$ with $xR \cap yR = 0$. We claim that $Fx, y$, the subalgebra of $R$ generated by $x$ and $y$, is free. For if not, we have a relation of the form $xf + yg + \alpha = 0$, where $f, g$ are nonzero "polynomials" in $x$ and $y$, and $\alpha \in F$. If $\alpha = 0$, then $xf = y(-g) \in xR \cap yR$, which is a contradiction. If $\alpha \neq 0$, then $xfy + ygy + \alpha y = 0$, so $xfy + y(gy + \alpha) = 0$. If $fy = 0$, then $f = 0$ since $R$ is a domain. We then get $yg + \alpha = 0$, so $y$ is invertible (and thus $yR = R$) which is a contradiction. Similarly, $gy + \alpha \neq 0$. Thus $x(fy) = y(gy + \alpha)$ is a nonzero element in $xR \cap yR$, again a contradiction. Hence $Fx, y$ is free. □

So far, we have only discussed the formation of *right* quotient rings; that is, in the overring, inverses appear on the right hand side. We could instead have done everything on the left, arriving at, for example, the definition of a *left Ore domain*, which is a domain $R$ satisfying $Ra \cap Rb \neq 0$ for all $a, b \in R$. In the favorable case where both left and right quotient rings exist, we shall see that they are naturally isomorphic, but for now let us give an example of a domain which is Ore on only one side.

**Example 5.6** (A right Ore domain which is not left Ore)**.** Let $F$ be a field, and let $\sigma : F \to F$ be a field endomorphism which is not onto. (eg. $t \mapsto t^2$, as a function from $\mathbb{R}(t)$ to itself) Let $R = F[x; \sigma]$ denote the "twisted polynomial ring" in the variable $x$. Elements of $R$ are *right* polynomials, by which we mean that coefficients are written on the right, but the variable $x$ does not commute with elements of $F$. Instead, we impose the relation

$$ax = x\sigma(a), a \in F.$$

Just as in the case of an ordinary polynomial ring, one can show that every right ideal of $R$ is principal, so $R$ is right noetherian, hence right Ore by the above theorem. But we claim that $R$ is not left Ore. To see this, let $c \in F$ be any element not in the image of $\sigma$. If it were the case that $Rx \cap Rxc \neq 0$, then we could find elements $b, d \in F$ such that

$$x^i bx = x^i dxc$$

which implies that $x^{i+1}\sigma(b) = x^{i+1}\sigma(d)c$, and thus that $\sigma(b) = \sigma(d)c$, and since $\sigma$ is an endomorphism, we get $c = \sigma(b)\sigma(d)^{-1} = \sigma(bd^{-1})$, a contradiction.

Note that we could instead take as $R$ the ring of *left* polynomials (with multiplication $xa = \sigma(a)x$), to get a left but not right Ore domain.

Application (Jategaonkar): The free algebra $F\langle x, y\rangle$ can be embedded into a division ring. As above, we take $R$ to be a right but not left Ore domain which is an algebra over the field $F$. Then $R$ contains $F\langle x, y\rangle$ as a subalgebra. Moreover, since $R$ is right Ore, it is contained in it's quotient division ring $Q$. Thus we have $F\langle x, y\rangle \subseteq R \subseteq Q$.

**Example 5.7.** The quotient ring of a noetherian ring needn't be artinian. We do this by constructing a non-artinian ring which is it's own classical quotient ring. Let $\mathbb{C}[[x, y]]$ denote the ring of formal power series in the variables $x$ and $y$ over $\mathbb{C}$, and let $R = \mathbb{C}[[x, y]]/\langle x^2, xy\rangle$. Let $\overline{x}$ and $\overline{y}$ denote the images of $x$ and $y$ in $R$. We note that $R$ is noetherian because $\mathbb{C}[[x, y]]$ is noetherian (analogously to the usual Hilbert basis argument), but $R$ is not artinian because $R/\langle\overline{x}\rangle \cong \mathbb{C}[[y]]$, which is not artinian since $y$ is a regular element of $\mathbb{C}[[y]]$ which is not invertible. Next, set $M = \langle\overline{x}, \overline{y}\rangle$, and note that $M$ is a maximal ideal of $R$ because $R/M \cong \mathbb{C}$. In fact, $M$ is the unique maximal ideal of $R$, because it contains every proper ideal of $R$. Now suppose that $a \in R$ is not invertible, so $aR \subsetneq R$. Thus $aR \subseteq M$, and since $\overline{x}M = 0$, we see that $\overline{x}aR = 0$, so $a$ is a zero divisor. Taking the contrapositive shows that every regular element of $R$ is a unit, so that $R$ is it's own classical ring of quotients.

We now head towards Goldie's Theorem, that a ring $R$ has a semisimple artinian ring of quotients iff $R$ is semiprime right Goldie. This will show, in particular, that semiprime right noetherian rings have quotient rings which are semisimple artinian. We begin with some definitions.

**Definition 5.8.** A right ideal $I$ of a ring $R$ is *essential* if $I \cap L \neq 0$ for every nonzero right ideal $L$ of $R$.

More generally, we call a submodule $N_R \subseteq M_R$ *essential (in $M_R$)* if $N \cap N' \neq 0$ whenever $N'$ is a nonzero submodule of $M$. Also, an $R$-module $M$ is *uniform* if every nonzero submodule is essential. That is, if the intersection of any two nonzero submodules is again nonzero.

Let $R$ be a right noetherian ring, and fix an element $a \in R$. Then we claim that there is some integer $n$ for which $a^n R \cap \operatorname{r.ann}(a) = 0$. To see this, note that we have an ascending chain of right ideals

$$\operatorname{r.ann}(a) \subseteq \operatorname{r.ann}(a^2) \subseteq \operatorname{r.ann}(a^3) \subseteq \cdots$$

which must terminate, since the ring is noetherian. So there is some $n$ for which $\operatorname{r.ann}(a^n) = \operatorname{r.ann}(a^{n+1}) = \cdots$. Now, if $s \in R$ is such that $a^n s \neq 0$ and $a^n s \in \operatorname{r.ann}(a)$, then $a^n s \in \operatorname{r.ann}(a^n)$, so $a^{2n} s = 0$. This implies that $s \in \operatorname{r.ann}(a^{2n}) = \operatorname{r.ann}(a^n)$, which is a contradiction.

**Definition 5.9.** Let $R$ be a ring, and set

$$S(R) = \{\, a \in R \mid \operatorname{r.ann}(a) \text{ is an essential right ideal of } R \,\},$$

the (right) *singular ideal of $R$.*

We need to justify this definition by showing that $S(R)$ is in fact an ideal of $R$. To see this, note that if $\operatorname{r.ann}(a_1)$ and $\operatorname{r.ann}(a_2)$ are essential right ideals of $R$, then so too is their intersection. Moreover, $\operatorname{r.ann}(a_1 + a_2) \supseteq \operatorname{r.ann}(a_1) \cap \operatorname{r.ann}(a_2)$, so $S(R)$ is closed under addition. Next, for $r \in R$, we have $\operatorname{r.ann}(ra) \supseteq \operatorname{r.ann}(a)$, so if $a \in S(R)$, so too is $ra$. Lastly, $ar \in S(R)$ because given any nonzero right ideal $I \lhd_r R$, if $rI = 0$, then $arI = 0$, so $I \subseteq \operatorname{r.ann}(ar)$. If $rI \neq 0$, then since $\operatorname{r.ann}(a)$ is essential, and since $rI \lhd_r R$ we have $rI \cap \operatorname{r.ann}(a) \neq 0$. We can then find some nonzero $ri \in rI \cap \operatorname{r.ann}(a)$. So $i \in \operatorname{r.ann}(ar) \cap I$, and hence $\operatorname{r.ann}(ar)$ is essential in $R$.

**Theorem 5.10.** *If $R$ is right noetherian, then $S(R)$ is a nilpotent ideal of $R$.*

*Proof.* Pick $a \in S(R)$. Then $\operatorname{r.ann}(a)$ is essential, and since we can find $n$ so that $a^n R \cap \operatorname{r.ann}(a) = 0$, we must have $a^n R = 0$. Since $R$ has a 1, this says that $a^n = 0$, so $S(R)$ is nil. Now, Levitzki's theorem says that a nil ideal of a right noetherian ring is nilpotent, so $S(R)$ is nilpotent. □

**Corollary 5.11.** *If $R$ is semiprime right noetherian, then $S(R) = 0$.*

We have introduced essential ideals because they play an integral role in Goldie's theorem, but we haven't yet shown how to actually produce an essential ideal. We do this now.

**Proposition 5.12.** *Let $R$ be any ring. If $I \lhd_r R$ is not essential, then there is some right ideal $K \lhd_r R$ such that $I \oplus K$ is essential.*

*Proof.* Use Zorn's Lemma to find a right ideal $K$ maximal with respect to $I \cap K = 0$. If $I \oplus K$ is not essential, then we can find a right ideal $U$ with $I \oplus K \cap U = 0$ which implies $I \cap K \oplus U = 0$ which contradicts our choice of $K$. □

Moreover, in a right noetherian ring, any regular element generates an essential right ideal. In fact, we only need a right regular element, this is the next

**Theorem 5.13.** *Let $R$ be right noetherian. If* $\mathrm{r.ann}(a) = 0$*, then $aR$ is essential.*

*Proof.* Choose $0 \neq I \lhd_r R$ and suppose that $aR \cap I = 0$. Then $aR \oplus I$ is a right ideal which properly contains $I$. Next, since $\mathrm{r.ann}(a) = 0$, left multiplication by $a$ is an injective $R$-module homomorphism of $R$, so preserves the direct sum. So $a^2R \oplus aI$ is also a direct sum, and of course we have $a^2R \oplus aI \subseteq aR$. Substituting for $aR$, we see that $a^2R \oplus aI \oplus I$ is also direct. Doing this again, we have $a^3R \oplus a^2I \oplus aI \oplus I$ is direct. Continuing, we produce an infinite direct sum of right ideals

$$I \oplus aI \oplus a^2I \oplus a^3I \oplus \cdots$$

which is impossible, since $R$ is noetherian. □

**Proposition 5.14.** *Let $R$ be semiprime right noetherian. Then every right regular element is also left regular.*

*Proof.* Suppose $\mathrm{r.ann}(a) = 0$, so that $aR$ is an essential right ideal. If $x \in \mathrm{l.ann}(a)$, then $\mathrm{r.ann}(x) \supseteq aR$, so $\mathrm{r.ann}(x) \neq 0$. Now, we can find $n$ such that $x^nR \cap \mathrm{r.ann}(x) = 0$, and since $\mathrm{r.ann}(x) \neq 0$, we conclude that $x^nR = 0$, so $x$ is nilpotent. This shows that $\mathrm{r.ann}(x)$ is a nil right ideal and hence nilpotent by Levitzki's theorem. This is a contradiction because $R$ is semiprime. □

We should note that in the proof of the last theorem, all that was needed was that $R$ did not contain an infinite direct sum of right ideals. This is, strictly speaking, a weaker condition than being right noetherian, and it motivates the following definition.

**Definition 5.15.** $R$ is *right Goldie* if (1) $R$ satisfies a.c.c. on right annihilator ideals, and (2) $R$ does not contain an infinite direct sum of right ideals.

At this point, one may reasonably demand an example of a ring which is right Goldie, but not right noetherian. Perhaps the easiest such is a polynomial ring in infinitely many variables over a field. It is an Ore domain (hence Goldie), but not noetherian.

**Theorem 5.16.** *Let $M$, $N$ be right $R$-modules, and let $\varphi : M \to N$ be an $R$-module homomorphism. If $L$ is essential in $N$, then $\varphi^{-1} = \{ m \in M \mid \varphi(m) \in L \}$ is essential in $M$.*

*Proof.* Let $B \subset M$ be a nonzero submodule of $M$. We need to show that $B \cap \varphi^{-1}(L) \neq 0$. Note that $\ker \varphi \subset \varphi^{-1}(L)$, so if $\varphi(B) = 0$, we're done. We may then assume that $\varphi(B) \neq 0$. Since $L$ is essential in $N$, we have $\varphi(B) \cap L \neq 0$, so we can find some $b \in B$ with the property that $0 \neq \varphi(b) \in L$. But then we have $0 \neq b \in B \cap \varphi^{-1}(L)$, which finishes the proof. $\square$

As a special case of the previous lemma, suppose that $I \lhd_r R$ is an essential right ideal of $R$. Fix an element $a \in I$, and note that *left* multiplication by $a$ is an $R$-module homomorphism from $R$ to $I$. It then follows that the inverse image of $I$ under this map is also an essential right ideal of $R$. For historical reasons, we denote this inverse image by $a^{-1}(I) := \{ r \in R \mid ar \in I \}$. This proves the following

**Theorem 5.17.** *If $I \lhd_r R$ is an essential right ideal of $R$, then for any $a \in R$, $a^{-1}(I) = \{ r \in R \mid ar \in I \}$ is essential in $R$.*

**Theorem 5.18** (Goldie)**.** *Let $R$ be semiprime right noetherian, and $I \lhd_r R$ a nonzero right ideal. Then there is some element $a \in I$ with $\mathrm{r.ann}(a) \cap I = 0$.*

*Proof.* We break the proof up into two cases, depending on whether or not $I$ is a uniform right ideal of $R$. So suppose first that $I$ is uniform. Note then that if $0 \neq K \lhd_r R$, with $K \subseteq I$, then for any $a \in I$, we have $a^{-1}(K) = \{ x \in R \mid ax \in K \}$ is an essential right ideal of $R$. Next, since $R$ is semiprime, $I^2 \neq 0$, so we can find nonzero elements $a, a' \in I$ with $aa' =\neq 0$. We will now show that this is the element $a$ we are looking for. If $\mathrm{r.ann}(a) \cap I = 0$ we're done. Otherwise, set $B = \mathrm{r.ann}(a) \cap I$. Then $B$ is essential in $I$ by uniformity, and so $(a')^{-1}(B)$ is essential in $R$. Now, if $x \in (a')^{-1}(B)$, then $aa'x = 0$, because $a'x \in \mathrm{r.ann}(a)$. This shows that $(a')^{-1}(B) \subseteq \mathrm{r.ann}(aa')$, so in particular $\mathrm{r.ann}(aa')$ is essential in $R$. This shows that $aa' \in S(R)$, the right singular ideal of $R$. Since $R$ is semiprime, $S(R) = 0$, so $aa' = 0$, a contradiction.

Now suppose that $I$ is not uniform, and choose a right ideal $I_0 \subseteq I$ maximal with respect to the property of containing an element $x \in I_0$ with $\mathrm{r.ann}(x) \cap I_0 = 0$. If $I_0 = I$ we're done. Now note that since $R$ has no infinite direct sums of right ideals, we can find a uniform right ideal $U \subseteq \mathrm{r.ann}(x) \cap I_0$. Choose such a $U$ which contains an element $u \in U$ with $\mathrm{r.ann}(u) \cap U = 0$. Next, note that we have $I_0 \subsetneq I_0 \oplus U$. Now, $x+u \in I_0 \oplus U$, and we will show that $\mathrm{r.ann}(x+u) \cap (I_0 \oplus U) = 0$, which will contradict our choice of $I_0$. Next, if $(x+u)(x'+u') = 0$, then $xx'+xu'+ux'+uu' = 0$. Since $U \subseteq \mathrm{r.ann}(x)$, this implies that $xu' = 0$, so we are left with $xx'+ux'+uu' = 0$. Since $xx' \in I_0$ and $ux'+uu' \in U$ and the sum is direct, we see that $xx' = 0$ and $ux' + uu' = 0$. Thus $x' = 0$ because $\mathrm{r.ann}(x) \cap I_0 = 0$. Also, $uu' = 0$ implies $u' = 0$ because $\mathrm{r.ann}(u) \cap U = 0$. This finishes the proof. $\square$

If $R$ is instead right Goldie (not necessarily right noetherian), we can modify the latter half of the proof by inducting on the Goldie rank of $R$. This would strengthen the previous lemma.

We have shown that in a semiprime right noetherian ring, a regular element generates an essential right ideal. The next lemma is essentially the converse.

**Theorem 5.19.** *Let $R$ be a semiprime right noetherian ring. Then $I \lhd_r R$ is essential iff $I$ contains a regular element.*

*Proof.* ($\Leftarrow$) Suppose $a \in I$ is regular, and suppose that $aR$ is not essential. Then there is some right ideal $K \lhd_r R$ with $aR \cap K = 0$. We want to show that $K + aK + \cdots + a^{i-1}K \cap a^iK = 0$ for all $i \in \mathbb{N}$. If not, then we can write

$$a^ik_i = k_0 + ak_1 + \cdots + a^{i-1}k_{i-1}$$

which implies that

$$k_0 = a[-(k_1 + ak_2 + \cdots + a^{i-1}k_{i-1}) + a^{i-1}k_i]$$

so $k_0 \in aK \cap K = 0$. Continuing in this fashion, we get $k_0 = k_1 = \ldots = k_{i-1} = 0$. This shows that we have an infinite direct sum of right ideal $K \oplus aK \oplus a^2K \oplus \cdots$, contradicting the fact that $R$ is noetherian. We thus conclude that $aR$ is essential, and since $aR \subseteq I$, $I$ is essential as well.

($\Rightarrow$) Now let $I \lhd_r R$ be essential. Since $R$ is semiprime right noetherian, $I$ is not nil. Thus there is $a_0 \in I$ which is not nilpotent, and without loss, we may assume that $\mathrm{r.ann}(a_0) = \mathrm{r.ann}(a_0^2)$. Suppose $\mathrm{r.ann}(a_0) \neq 0$. Then since $I$ is essential, we have $\mathrm{r.ann}(a_0) \cap I \neq 0$, which implies that $\mathrm{r.ann}(a_0) \cap I$ is not nil, so we can find $a_1 \in \mathrm{r.ann}(a_0) \cap I$ with $a_1$ not nilpotent, and again without loss we may assume that $\mathrm{r.ann}(a_1) = \mathrm{r.ann}(a_1^2)$. If $\mathrm{r.ann}(a_1) \neq 0$, repeat this argument to produce a sequence of elements $\{a_i\}$ with $a_i \in \mathrm{r.ann}(a_0) \cap \mathrm{r.ann}(a_1) \cap \cdots \cap \mathrm{r.ann}(a_{i-1})$, and $\mathrm{r.ann}(a_i) = \mathrm{r.ann}(a_i^2)$. Now, we claim that $(a_0R + \ldots + a_{i-1}R) \cap a_iR = 0$ for all $i$. For otherwise, we can write $a_i r_i = \sum_{j<i} a_j r_j$, which implies that $a_0^2 r_0 = a_1^2 r_1 = \cdots = a_{i-1}^2 r_{i-1}$ since $a_i \in \operatorname{r.ann}(a_j)$ for $j < i$. We then conclude that $r_0 \in \operatorname{r.ann}(a_0^2) = \operatorname{r.ann}(a_0)$, and similarly $r_j \in \operatorname{r.ann}(a_j)$ for $j < i$. Thus $a_i r_i = 0$. Now, since $R$ is right noetherian, it cannot contain an infinite direct sums of right ideals, so the process of producing these $a_i$ must terminate. Hence there is some $n$ for which $\operatorname{r.ann}(a_0) \cap \cdots \cap \operatorname{r.ann}(a_n) = 0$. Then $a = a_0 + \ldots + a_n$ is right regular, as $ax = 0 \implies a_i a x = a_i^2 x = 0$, so $x \in \operatorname{r.ann}(a_0) \cap \cdots \cap \operatorname{r.ann}(a_n)$. □

In fact, the hypothesis that $R$ be right noetherian can be weakened to right Goldie, and the proof can be easily modified to prove the following.

**Theorem 5.20.** *Let $R$ be semiprime right Goldie, and $I \lhd_r R$. Then $I$ is essential in $R$ iff $I$ contains a (right) regular element.*

*Proof.* Left to the reader □

Now, suppose that $R$ is semiprime right Goldie, and choose $a, c \in R$ regular. Then $cR$ is an essential right ideal of $R$, and so $a^{-1}(cR)$ is also essential. By the last lemma, $a^{-1}(cR)$ contains a regular element, say $b$, and we have $ab = cd$ for some $d \in R$. This is precisely the Ore condition, so we have shown the following

**Theorem 5.21.** *If $R$ is semiprime right Goldie, then $S = \{$ regular elements of $R$ $\}$ is right Ore, and so $R$ has a right classical ring of quotients.*

**Corollary 5.22.** *If $R$ is semiprime right noetherian, then $R$ has a right classical quotient ring.*

5.1. **Ideals in $Q(R)$.** Having shown that semiprime right Goldie rings have rings of quotients, we next want to study these resulting quotient rings. We will see that the quotient ring of a semiprime right Goldie ring is semisimple artinian (and hence a finite direct sum of matrix rings over division rings). First we start by relating the (right) ideal structures of a ring $R$ and it's quotient ring $Q(R)$.

Throughout this section, $R$ is a ring with classical quotient ring $Q = Q(R)$ (so all regular elements of $R$ are invertible in the overring $Q$). The first lemma shows that we can find "common denominators".

**Theorem 5.23.** *If $d_1, \ldots, d_t$ are regular elements of $R$, then there is a regular element $d \in R$, and elements $c_1, \ldots, c_t$ in $R$ such that $d_i^{-1} = c_i d^{-1}$ for all $i$.*

*Proof.* □

Recall that $\mathcal{S}(R)$ denotes the set of regular elements of $R$. The previous lemma immediately implies the following

**Corollary 5.24.** *If $I \lhd_r R$, then $IQ = \{ id^{-1} \mid i \in I, d \in \mathcal{S}(R) \}$*

**Theorem 5.25.** *If $K \lhd_r Q$, then $(K \cap R)Q = K$.*

*Proof.* Given $k \in K$, we can write $k = ac^{-1}$ for some $a, c \in R$, $c$ regular. Then $kc \in K \cap R$, and hence $k = (kc)c^{-1} \in (K \cap R)Q$. The reverse containment is obvious. □

**Theorem 5.26.** *If $I_1 \oplus \cdots I_n$ is a direct sum of right ideals of $R$, then $I_1Q \oplus \cdots I_nQ = (I_1 \oplus \cdots I_n)Q$ is a direct sum of right ideals of $Q$.*

*Proof.* For the equality then $I_1Q \oplus \cdots I_nQ = (I_1 \oplus \cdots I_n)Q$, just find common denominators. That the sum is direct is a consequence of the previous lemma. □

Recall that given a right ideal in any ring, we can find a "complementary" right ideal such that the direct sum is essential. If the ring is $Q$, the quotient ring of a semiprime right Goldie ring $R$, then the complement can actually be chosen so that the direct sum is all of $Q$. This is the next

**Theorem 5.27.** *Let $K \lhd_r Q$. Then there is a right ideal $K' \lhd_r Q$ such that $K \oplus K' = Q$.*

*Proof.* Consider $(K \cap R) \lhd_r R$. We can then find $I \lhd_r R$ such that $(K \cap R) \oplus I$ is essential in $R$, hence contains a regular element. Then $((K \cap R) \oplus I)Q = Q$ because every regular element of $R$ is a unit in $Q$. Thus $K \oplus IQ = (K \cap R)Q \oplus IQ = Q$. □

As a result, we can now show that $Q$ is semisimple artinian. Choose any right ideal $K \lhd_r Q$, and write $Q = K \oplus K'$. Then $1 = e + (1-e)$ with $e \in K$, $(1-e) \in K'$. Clearly $e^2 = e$, so $K = eQ$, which shows that every right ideal is generated by an idempotent. Hence $Q$ is right noetherian (since right ideals are f.g.) Moreover, this shows that $Q$ has no nonzero nil (right) ideals. For $0 \neq e$ idempotent implies that $e$ is not nilpotent. Thus $Q$ has no nonzero nilpotent (right) ideals, so $Q$ is semiprime. All that remains is to show that $Q$ is right artinian. If this is not the case, then we can find a strictly descending chain of idempotent generated right ideals

$$e_1Q \supsetneq e_2Q \supsetneq e_3Q \supsetneq \cdots .$$

By a previous lemma, given any right ideal $I$ of $Q$, we can find a complement $K$ such that $I \oplus K = Q$. By taking intersections with $e_1Q$, we can then find a right ideal $U_2 \lhd_r Q$ such that $e_1Q = U_2 \oplus e_2Q$. Similarly, we can write $e_2Q = U_3 \oplus e_3Q$ for some right ideal $U_3 \lhd_r Q$. Continuing this process yields an infinite direct sum

$$U_2 \oplus U_3 \oplus \cdots ,$$

which is a contradiction, because $Q$ has no infinite direct sums of right ideals. Thus $Q$ is also right artinian (and hence artinian by Wedderburn).

So we have shown that if $R$ is semiprime right Goldie, then $Q$ is semisimple artinian. If in fact $R$ is prime right Goldie, then one may expect the quotient ring to be simple artinian. We can now show that this is in fact the case

**Theorem 5.28.** *Let $R$ be prime right Goldie. Then $Q$ is simple artinian.*

*Proof.* We already know that $Q$ is artinian, so we just need to show that $Q$ has no nontrivial proper ideals. If $0 \neq I \lhd Q$, then $0 \neq (I \cap R) \lhd R$. In a prime ring, every nonzero ideal is essential as a right ideal (check!), so we know that $I \cap R$ contains a regular element of $R$. Since regular elements of $R$ become invertible in $Q$, we then have $I = (I \cap R)Q = Q$. □

We wish to prove the converse to Goldie's theorem as well, but first we need a digression in order to discuss annihilator ideals in more detail.

**Proposition 5.29.** *Let $X$ be any subset of a ring $R$, and set $K = \mathrm{r.ann}(X) = \{ r \in R \mid Xr = 0 \}$. Then $\mathrm{r.ann}(\mathrm{l.ann}(K)) = K$.*

*Proof.* Clearly $K \subseteq \mathrm{r.ann}(\mathrm{l.ann}(K))$, because $\mathrm{l.ann}(K)K = 0$. Now, suppose that $r \in \mathrm{r.ann}(\mathrm{l.ann}(K))$, so $\mathrm{l.ann}(K)r = 0$. Since $XK = 0$, we have $X \subseteq \mathrm{l.ann}(K)$, so $Xr = 0$, ie $x \in K$. □

This proposition says that there is a 1-1, order reversing correspondence between right and left annihilator ideals of $R$. As a consequence, we get the following

**Corollary 5.30.** *$R$ satisfies a.c.c. on right annihilators iff $R$ satisfies d.c.c. on left annihilators.*

Moreover, if $T \subseteq R$ is a subring of $R$, and $X$ is any subset of $R$, then $\mathrm{r.ann}_T(X) = \mathrm{r.ann}_R(X) \cap T$, which shows that $T$ inherits chain conditions on annihilators from $R$. In other words, if $R$ satisfies a.c.c. or d.c.c. on right (resp. left) annihilators, then so does $T$. (Contrast this with the fact that subrings needn't inherit chain conditions on one sided ideals, ie a subring of an artinian ring needn't be artinian).

**Example 5.31.** A ring which satisfies a.c.c. on right annihilators, but not on left annihilators. Let $\mathbb{A}_1$ denote the first Weyl algebra (over $\mathbb{C}$)

and let $I$ be any nonzero right ideal of $\mathbb{A}_1$. Then $\mathbb{A}_1/I$ is a $(\mathbb{C}, \mathbb{A}_1)$-bimodule under usual multiplication, and we can form a ring

$$R = \begin{pmatrix} \mathbb{C} & \mathbb{A}_1/I \\ 0 & 0 \end{pmatrix}.$$

We will show that $R$ satisfies a.c.c. on right annihilators, but doesn't satisfy d.c.c. on right annihilators, so doesn't satisfy a.c.c. on left annihilators.

Since $\mathbb{A}_1$ is a domain, $I$ is an essential right ideal, so contains a nonzero regular element, say $c$. Since $\mathbb{A}_1$ is right Ore, given any $a \in \mathbb{A}_1$, we can write $ab = cd$ for some $b, d \in \mathbb{A}_1$. Thus $(a+I)b = I$. This shows that any finite subset of $\mathbb{A}_1/I$ has a nonzero right annihilator (ie $\mathbb{A}_1/I$ is a torsion $\mathbb{A}_1$-module). We can then lift this to $R$ in the obvious way. If $X \subset \mathbb{A}_1/I$ is any finite subset, then

$$\operatorname{r.ann}_R(\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} \mathbb{C} & \mathbb{A}_1/I \\ 0 & \operatorname{r.ann}_{\mathbb{A}_1}(X) \end{pmatrix}$$

Now, suppose that $R$ satisfied d.c.c. on right annihilators, and let $M$ be a minimal element in the set of right annihilators of finite subsets of $\mathbb{A}_1/I$. So $M = \operatorname{r.ann}_{\mathbb{A}_1}(v_1 + I, \ldots, v_n + I)$ for some elements $v_i \in A_1$. Now we claim that $(\mathbb{A}_1/I)M = 0$. For choose any $a \in \mathbb{A}_1$, and note that $\operatorname{r.ann}_{\mathbb{A}_1}(v_1 + I, \ldots, v_n + I, a + I) \subseteq M$, so this right annihilator equals $M$ by minimality. This shows that $M$ annihilates all of $A_1/I$, so $\operatorname{r.ann}_{\mathbb{A}_1}(\mathbb{A}_1/I) \neq 0$. But since $\mathbb{A}_1/I$ is a right $\mathbb{A}_1$-module, it's right annihilator is a two-sided ideal of $\mathbb{A}_1$, and since $\mathbb{A}_1$ is simple, we have $(\mathbb{A}_1/I)\mathbb{A}_1 = 0$. This is clearly a contradiction.

Note that as an added bonus, we see that this ring $R$ cannot be embedded into an artinian ring (for then it would inherit d.c.c. on right annihilators from the artinian overring).

We can now get back to Goldie's theorem. We want to show that the converse to Goldie's theorem is also true, which we do now.

**Theorem 5.32.** *If $Q$ is semisimple artinian, then $R$ is semiprime right Goldie.*

*Proof.* First we need to show that $R$ is right Goldie. This follows easily from the correspondence between right ideals of $Q$ and right ideals of $R$. Next, suppose that $0 \neq N \lhd R$ satisfies $N^2 = 0$. Then $NQ \lhd_r Q$. (note that $(NQ)^2$ needn't be 0). If $NQ = Q$, then $NQ$ contains a unit, and since $R$ is an order in $Q$, $N$ contains a regular element of $R$. This is a contradiction, so $NQ \subsetneq Q$. Consider $\operatorname{l.ann}_R(N) = \{ r \in R \mid rN = 0 \}$. We will show that $\operatorname{l.ann}_R(N)$ is an essential right ideal of $R$. To see this, let $0 \neq I \lhd_r R$. If $IN = 0$, then $I \subseteq \operatorname{l.ann}_R(N)$ and we're done.

Otherwise, $0 \neq IN \subseteq (I \cap N)$, so $(IN)N = 0$, and we can conclude that $IN \subseteq (I \cap \operatorname{l.ann}_R(N))$. Thus $\operatorname{l.ann}_R(N)$ is essential, so contains a regular element, say $c$. But then $cN = 0$, and since $c$ is regular, we can cancel to get $N = 0$, a contradiction. This shows that $R$ is semiprime. $\square$

Summarizing, we have proved all of Goldie's theorem, which we now state as a single result.

**Theorem 5.33** (Goldie)**.** *A ring $R$ has a classical right quotient ring $Q$ which is semisimple artinian iff $R$ is semiprime right Goldie.*

We now offer an application, which was an open problem for some time until Goldie's seminal work. We will assume (as was known at the time) that nil subrings of semisimple artinian rings are actually nilpotent (the proof is left as an exercise).

**Theorem 5.34.** *Let $R$ be a right noetherian ring. If $N$ is a nil subring, then $N$ is nilpotent.*

*Proof.* Let $P = P(R)$ denote the nil radical of $R$, which is the unique maximal nilpotent ideal of $R$ by Levitzki's theorem. If $N \subseteq P$, we're done. Otherwise, $\overline{N} = N/(P \cap N)$ is a nil subring of $\overline{R} = R/P$. Since $\overline{R}$ is semiprime right noetherian, it has a quotient ring $Q$ which is semisimple artinian. Hence $\overline{N}$ is nilpotent, so some power of $N$ lies in $P$. But since $P$ is also nilpotent, taking a large enough power of $N$ shows that $N$ is nilpotent as well. $\square$

**Definition 5.35.** An ideal $I \lhd R$ is called *prime* if whenever $A, B \lhd R$ with $AB \subseteq I$, we must have either $A \subseteq I$ or $B \subseteq I$. We call a ring $R$ prime if 0 is a prime ideal.

Recall that we call a right artinian ring *semisimple* if it has no nonzero nilpotent ideals. In fact, one can define the notion of semisimplicity for more general rings (ie. not necessarily artinian), which we do now.

**Definition 5.36.** A ring $R$ is called *semisimple* if every right ideal is a direct summand of $R_R$. That is, given a right ideal $I$ of $R$, there is some right ideal $J$ such that $R = I \oplus J$.

**Remark 5.37.** Note that $R$ is prime iff whenever $A, B$ are nonzero ideals of $R$, then $AB \neq 0$.

**Theorem 5.38.** *If $R$ is prime, then every nonzero ideal of $R$ is essential.*

*Proof.* If $I, J$ are nonzero ideals of $R$, then so too is $IJ$, which is contained in $I \cap J$. □

**Proposition 5.39.** *Let $R$ be a prime ring, $0 \neq L \lhd_l R$ and $K \lhd_r R$. Then $L \cap K \neq 0$.*

*Proof.* $KL \subseteq L \cap K$, so if $L \cap K = 0$ then $KL = 0$, which implies that $(RK)(LR) = 0$. But $RK$ and $LR$ are nonzero two-sided ideals of $R$, a contradiction. □

**Corollary 5.40.** *If $R$ is a prime ring with zero divisors, then $R$ has nonzero nilpotent elements.*

*Proof.* Let $0 \neq a, b \in R$ with $ab = 0$. Then $0 \neq Ra \lhd_l R$ and $0 \neq bR \lhd_r R$, so from above we have $Ra \cap bR \neq 0$. We can then find $s, t \in R$ with $sa = bt$, so $(sa)^2 = sa(bt) = 0$. □

## 6. Division Algebras

Let $\mathbb{H}$ denote the algebra of (real) quaternions, and recall that $Z(\mathbb{H}) = \mathbb{R}$, so $\dim_{\mathbb{R}} \mathbb{H} = 4$, a perfect square. If we start considering other division algebras, we will soon see that they always have dimensions over their centers which are perfect squares. In fact this is no accident, as we will see that division algebras are intimately related to matrix algebras (over the larger fields), which will allow us to show prove this somewhat mysterious observation on dimensions. We start with a definition.

**Definition 6.1.** An algebra $A$ is a *central simple algebra (over a field $F$)* if $A$ is simple and $Z(A) = F$. We sometimes abbreviate this by saying $A$ is C-S.

**Example 6.2.**

(1) Any field is central simple over itself.
(2) $M_n(F)$ is central simple over $F$.
(3) $\mathbb{H}$ is central simple over $\mathbb{R}$.
(4) Any division ring is central simple over its center, which is a field.
(5) $\mathbb{A}_1(\mathbb{C})$, the first Weyl algebra, is central simple over $\mathbb{C}$ (but not over $\mathbb{R}$).

The main reason we are interested in central simple algebras is because they have nice properties with respect to tensor products.

**Proposition 6.3.** *If $A$ is central simple over $F$ and $B$ is any simple $F$-algebra, then $A \otimes_F B$ is central simple over $F$.*

*Proof.* In this proof, tensor products are always over $F$, so we drop the subscript. Let $U \lhd A \otimes B$ be a nonzero ideal, and choose $u = \sum a_i \otimes b_i \in U$ so that the $b_i$ are linearly independent over $F$. Define the *length* of $u$ to be the minimal number of $a_i$ which appear in such an expression, and choose $u$ of minimal length in $U$. Now, if $r, s \in A$, then

$$(r \otimes 1)u(s \otimes 1) = \sum_{i=1}^{m} (ra_i s) \otimes b_i \in U.$$

Now, since $A$ is simple, given any $0 \neq a \in A$, we have $AaA = A$. By choosing appropriate $r, s \in A$, we can find $u_1 \in U$, also of minimal length, so that $u_1 = 1 \otimes b_1 + a_2' \otimes b_2 + \cdots + a_m' \otimes b_m$. Now, let $a \in A$ be arbitrary, and note that $[a \otimes 1, u_1] := (a \otimes 1)u - u(a \otimes 1) \in U$ has shorter length than $u_1$ because $[a \otimes 1, 1 \otimes b_1] = 0$. Thus $[a \otimes 1, u_1] = 0$ by the minimality of the length of $u_1$. Thus $[a, a_2'] \otimes b_2 + [a, a_3'] \otimes b_3 + \cdots + [a, a_m'] \otimes b_m = 0$, and so $[a, a_i'] = 0$ for all $i$ because the $b_i$ are linearly independent. This shows that all the $a_i'$ are actually in $Z(A) = F$. Thus $u_1 = 1 \otimes (b_1 + a_2' b_2 + \cdots + a_m' b_m) = 1 \otimes b$ for some $b \in B$. Now, since $B$ is simple, $BbB = B$, so $U \supseteq (1 \otimes B)(1 \otimes b)(1 \otimes B) = 1 \otimes B$. Thus $U \supseteq (A \otimes 1)(1 \otimes B) = A \otimes B$, so $A \otimes B$ is simple. □

**Corollary 6.4.** *If $A$ and $B$ are both central simple over $F$, then $A \otimes_F B$ is central simple over $F$ as well.*

*Proof.* We just need to show that $Z(A \otimes B) = F$. As in the previous proof, choose $z = \sum a_i \otimes b_i \in Z(A \otimes B)$ with the $b_i$ linearly independent. Then for any $a \in A$, we have

$$0 = [z, a \otimes 1] = \sum [a_i, a] \otimes b_1$$

and thus $[a_i, a] = 0$ for all $i$ by linear independence of the $b_i$. Thus all the $a_i \in F$. We can then rewrite $z$ as $z = 1 \otimes b$ for some $b \in B$. Commuting this with $1 \otimes b'$ shows that $b \in F$ as well. Hence $Z(A \otimes B) = F \otimes_F F \cong F$, as desired. □

**Remark 6.5.** That $A$ above was actually central simple is crucial. In general nothing can be said about the tensor product of two simple algebras. For example, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not even a domain.

Note that any division ring is a central simple algebra over it's center.

**Theorem 6.6.** *If $D$ is a finite dimensional division algebra such that $\dim_{Z(D)}(D) = m < \infty$, then $m$ is a perfect square.*

*Proof.* Let $Z = Z(D)$, and let $\overline{Z}$ denote the algebraic closure of $Z$. Consider $D \otimes_Z \overline{Z}$ an algebra over $\overline{Z}$. Recall that $\dim_{\overline{Z}}(D \otimes_Z \overline{Z}) = \dim_Z(D)$. Moreover, $D \otimes_Z \overline{Z}$ is actually central simple over $\overline{Z}$. Now, by the Weddderburn-Artin theorem, $D \otimes_Z \overline{Z} \cong M_t(\Delta)$ for some division ring $\Delta$. But since $\overline{Z}$ is algebraically closed, and since $\Delta$ is an algebra over $\overline{Z}$, we have $\Delta = \overline{Z}$, so $D \otimes_Z \overline{Z} \cong M_t(\overline{Z})$. Taking dimensions yields

$$m = \dim_Z(D) = \dim_{\overline{Z}}(D \otimes_Z \overline{Z}) = \dim_{\overline{Z}}(M_t(\overline{Z})) = t^2.$$

$\square$

Recall that if $R$ is a ring, then $R^{op}$ denotes the *opposite ring.* This is a ring whose underlying group structure is the same as $R$, but with the order of multiplication reversed. That is, if we denote elements of $R^{op}$ by $r^{op}$, then we have $a^{op}b^{op} = (ba)^{op}$. Of course, every ring $R$ is anti-isomorphic to $R^{op}$, and if $R$ is commutative, then $R \cong R^{op}$. This can happen in the noncommutative case as well, for example the transpose operator on $M_2(F)$ is an involution and it follows that $M_2(F) \cong M_2(F)^{op}$. As another example, $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is not isomorphic to its opposite because it is only noetherian on one side. (In fact, $R \cong R^{op}$ iff $R$ has an involution, but we won't prove this).

We note that if $M$ is a left $R$-module, then $M$ is also naturally a right $R^{op}$-module via

$$m \cdot a^{op} := am$$

**Theorem 6.7.** *If $A$ is a finite dimensional central simple algebra over $F$, then $A \otimes_F A^{op} \cong M_n(F)$, where $n = \dim_F(A)$.*

*Proof.* Consider $\mathrm{End}_F(A)$, the ring of all $F$-linear transformations on the vector space $A$. We consider $A$ as a $(A, A^{op})$-bimodule using left and right multiplication. More precisely, give $a \in A$, let $\lambda_a$ and $\rho_a$ denote left and right multiplication respectively, by $a$. If we define

$$A_\lambda = \{\, \lambda_a \mid a \in A \,\} \text{ and } A_\rho = \{\, \rho_a \mid a \in A \,\}$$

Then we naturally have $A_\lambda \cong A$ and $A_\rho \cong A^{op}$ as rings. Next note that $A_\lambda$ and $A_\rho$ commute element-wise by associativity of $A$. If we denote the subalgebra of $\mathrm{End}_F(A)$ generated by $A_\lambda$ and $A_\rho$ by $A_\lambda A_\rho$, then we get a ring homomorphism

$$\phi : A \otimes A^{op} \to A_\lambda A_\rho \text{ defined by } \sum a_i \otimes b_i^{op} \to \sum \lambda_{a_i} \rho_{b_i}.$$

Since $A \otimes A^{op}$ is simple, $\phi$ is injective. Moreover, $\phi$ is certainly a surjection onto $A_\lambda A_\rho$. By counting dimensions, we have $\dim_F(A_\lambda A_\rho) = \dim_F(A \otimes A^{op}) = n^2$, so $A_\lambda A_\rho = \mathrm{End}_F(A) \cong M_n(F)$. $\square$

### 6.1. **Brauer Groups.**

**Definition 6.8.** If $A$ and $B$ are finite dimensional central simple $F$-algebras, then we say $A$ is equivalent to $B$, written $A \sim B$, if there exist integers $m, n$ such that $A \otimes_F M_n(F) \cong B \otimes_F M_m(F)$.

One can check that this is indeed an equivalence relation on the set of finite dimensional central simple $F$-algebras. We write $[A]$ for the equivalence class of $A$. We can define a multiplication on equivalence classes via

$$[A][B] = [A \otimes_F B].$$

Under this multiplication, we have $[A][F] = [F][A] = [A]$ for any finite dimensional central simple $F$-algebra $A$. Moreover, since $A \otimes_F A^{op} \cong M_n(F)$ for some $n$, we have

$$[A][A^{op}] = [A^{op}][A] = [F]$$

Thus the collection of equivalence classes so defined forms a group, called the *Brauer Group of* $F$, which we denote by $Br(F)$. The Brauer group plays a fundamental role in the study of division algebras, representation theory, etc. In general, computing the Brauer group of a given field is difficult, but we can do so in a few simple cases:

**Example 6.9.**

- $Br(\mathbb{C})$ is trivial, because the only finite dimensional central simple algebras over $\mathbb{C}$ are matrix rings over $\mathbb{C}$. Recall this is because $\mathbb{C}$ is algebraically closed, so there are no finite dimensional division algebras over $\mathbb{C}$.
- $Br(\mathbb{R}) \cong \mathbb{Z}_2$. This follows because other than $\mathbb{R}$ itself, the only finite dimensional division algebras over $\mathbb{R}$ are $\mathbb{C}$ and $\mathbb{H}$.
- A theorem of Tsen shows that $Br(\mathbb{C}(t))$ is trivial (Tsen's theorem is difficult though).

Next we want to prove a theorem about maximal subfields of division algebras. First, note that $\mathbb{H}$ is a central simple division algebra over $\mathbb{R}$, and $\mathbb{C} \subset \mathbb{H}$ is a maximal subfield. We see that $\dim_{\mathbb{R}}(\mathbb{C}) = 2$, and $\dim_{\mathbb{R}}(\mathbb{H}) = 2^2$. This is no accident, as the following theorem shows.

**Theorem 6.10.** *Let $D$ be a finite dimensional division algebra over $F = Z(D)$, and let $K$ be a maximal subfield of $D$. Then $D \otimes_F K \cong M_n(K)$, where $n = \dim_F(K)$, and $\dim_F(D) = n^2$.*

*Proof.* We consider $D$ as a vector space over $F$, and we let $D$ and $K$ act on $D$ via left and right multiplication, respectively. This gives us a way of viewing both $D$ and $K$ as subalgebras of $\mathrm{End}_F(D)$. To avoid confusion, we write $D_\lambda$ and $K_\rho$ for the images of $D$ and $K$ respectively in $\mathrm{End}_F(D)$. (The notation is as in the proof of the previous lemma) Morover, these two actions commute elementwise because $D$ is associative. Let $D_\lambda K_\rho$ denote the (noncommutative) subalgebra of $\mathrm{End}_F(D)$ generated by $D_\lambda$ and $K_\rho$. Since $D$ is naturally a left $\mathrm{End}_F(D)$-module, $D$ is also a leeft $D_\lambda K_\rho$-module. As such, $D$ is irreducible, because $D_\lambda$ acts transitively on $D$. Next we claim that the centralizer of $D_\lambda K_\rho$ in $\mathrm{End}_F(D)$ is just $K_\lambda$. That is, $\mathrm{End}_{D_\lambda K_\rho}(D) = K_\rho$. Clearly $K_\rho$ commutes with $D_\lambda K_\rho$. Also, since $K$ is a maximal subfield, the centralizer in $D$ of $K$ is precisely $K$. It follows that $\mathrm{End}_{D_\lambda K_\rho}(D) = K_\rho$. We then see that $D_{lambda} K_\rho \cong M_t(K)$ for some $t$. Now, we have a surjective ring homomorphism from $D \otimes_F K$ to $D_\lambda K_\rho$ which is also injective because $D \otimes_F K$ is simple. Computing dimensions, we see that

$$t^2 = \dim_K(M_t(K)) = \dim_K(D_\lambda K_\rho) = \dim_K(D \otimes_F K) = \dim_F(D) = n^2$$

□

As an application, we can answer the following question. Suppose $D$ is a division ring with center $Z$, and $D'$ is a subdivision ring with center $Z'$. If $D$ is finite dimensional over $Z$, must $D'$ be finite dimensional over $Z'$? This was an outstanding problem for some time until the development of PI theory. The answer is yet, and it can be obtained from a result of Kaplansky about PI rings, but we can now present an easier proof.

First note that $ZZ'$ is a subfield of $D$. Since $D'$ is central simple over $Z'$, we know that $D' \otimes_{Z'} ZZ'$ is simple. Moreover, $\dim_{Z'}(D') = \dim_{ZZ'}(D' \otimes_{Z'} ZZ')$, so it is enough to show that $D' \otimes_{Z'} ZZ'$ is finite dimensional over $ZZ'$. As we have been doing, we get a surjection from $D' \otimes_{Z'} ZZ'$ onto $D(ZZ')$, which must be an isomorphism since the former is simple. Now, since $D(ZZ') \subseteq D$, and since $D$ is finite dimensional over $Z$, $D(ZZ')$ is finite dimensional over $Z$ as well, and hence also finite dimensional over $Z' \supseteq Z$. This completes the proof.

We turn now to the celebrated Skolem-Noether theorem, which gives sufficient conditions under which certain ring isomorphisms are given by conjugation. In particular, we will see that every automorphism of a simple artinian ring is inner.

**Theorem 6.11** (Skolem-Noether)**.** *Let $R$ be a simple artinian ring with center $F$, a field. Let $A$ and $B$ be $F$-subalgebras of $R$, and $\varphi : A \to B$ an $F$-algebra isomorphism (so $\varphi$ fixes $F$ element-wise). Then there exists $x \in R$ such that $\varphi(a) = x^{-1}ax$ for all $a \in A$*

Before we prove this, let us give some applications to division rings.

**Theorem 6.12** (Wedderburn)**.** *A finite division ring is a field.*

*Proof.* Let $D$ be a finite division ring with center $Z$, and let $K$ be a maximal subfield of $D$. We then know that $[D : Z] = n^2$ for some $n$, and $[K : Z] = n$. Since, up to isomorphism, there is but one finite field of a given cardinality, we have that any two maximal subfields of $D$ are isomorphic, and moreover such an isomorphism fixes $Z$ element-wise. Now, if $K'$ is any other maximal subfield, then by Skolem-Noether, there is some $x \in D$ such that $K' = x^{-1}Kx$. Also, given any $a \in D$, $Z(a)$ is contained in a maximal subfield of $D$. Set $D^* = D - \{0\}$, so by the previous two remarks, we have

$$D^* = \cup_{x \neq 0} x^{-1} K^* x$$

where $K^* = K - \{0\}$. But $D^*$ is a finite group, and a finite group cannot be the union of the conjugates of a proper subgroup. Hence we must conclude that $K^* = D^*$, so $K = D$ is a field. □

**Theorem 6.13** (Frobenius)**.** *The only noncommutative finite dimensional central simple division algebra over* $\mathbb{R}$ *is* $\mathbb{H}$.

*Proof.* Let $D$ be a division ring with center $\mathbb{R}$, and let $K$ be a maximal subfield of $D$. Then $K$ is a finite dimensional field extension of $\mathbb{R}$, so either $K = \mathbb{R}$, or $K \cong \mathbb{C}$. By what should be a routine argument at this point, we have $[D : \mathbb{R}] = [K : \mathbb{R}]^2$, and since $D$ is not commutative, $[K : \mathbb{R}] = 2$, ie $K \cong \mathbb{C}$. So $[D : \mathbb{R}] = 4$, and $D$ contains $\mathbb{C}$ as a maximal subfield. Note that conjugation is an $\mathbb{R}$-algebra automorphism of $\mathbb{C}$, so by Skolem-Noether, we can find $x \in D$ such that $x^{-1}(a + bi)x = a - bi$ for any $a + bi \in \mathbb{C}$. In particular, we get $x^{-1}ix = -i$, so $ix^2 = x^2 i$. This shows that $x^2$ commutes element-wise with $\mathbb{C}$, hence $x^2 \in \mathbb{C}$ by maximality of $\mathbb{C}$. We claim that in fact $x^2 \in \mathbb{R}$. If not, then $\mathbb{R}(x^2)$ is a degree 2 extension of $\mathbb{R}$, contained in $\mathbb{C}$, hence $\mathbb{R}(x^2) = \mathbb{C}$. Now note that $x$ commutes with $x^2$, so by maximality of $\mathbb{C}$, $x \in \mathbb{C}$. But this is a contradiction, because $x$ doesn't commute with $i$. Thus we can conclude that $x^2 \in \mathbb{R}$. Moreover, $x^2 \in \mathbb{R}_{<0}$, because if $x^2$ were positive, then $x \in \mathbb{R}$, again a contradiction because $x$ doesn't centralize $i$. We can then write $x^2 = -\alpha^2$ for some $\alpha \in \mathbb{R}$. Set $j = x/\alpha$. Then $j^2 = -1$, and $ji = -ij$. If we then consider the subalgebra of $D$ generated by $\{1, i, j, ij\}$, we see that this subalgebra is isomorphic to $\mathbb{H}$, and thus $D \cong \mathbb{H}$ as desired. □

We next give a generalization, due to Jacobson, of Wedderburn's theorem on finite division rings.

**Theorem 6.14** (Jacobson)**.** *A division ring, algebraic over a finite field, is commtative.*

*Proof.* Let $F$ be a finite field, and $D$ a division algebra which is algebraic over $F$. (Note that $F \subseteq Z(D)$). We wish to show that $Z(D) = D$, so suppose there is some element $x \in D - Z(D)$. Since $x$ is algebraic over $F$, the field extension $F(x)$ is finite dimensional over $F$, and hence Galois. Thus there is some nontrivial field automorphism $\varphi : F(x) \to F(x)$ which fixes $F$ element-wise. Since $\varphi$ is nontrivial, $\varphi(x) = x^r$ for some $r > 1$. By Skolem-Noether, $\varphi(x) = y^{-1}xy$ for some $y \in D$, so we have

$$y^{-1}xy = x^r \text{ or } xy = yx^r.$$

Let $A$ denote the subalgebra of $D$ generated by $x$ and $y$. Since $A \subseteq D$, $A$ is a domain, algebraic over $F$, hence is a division ring. Now, we have $[A : F(x)] < \infty$ because $y$ is algebraic over $F$, hence also over $F(x)$. We already saw that $[F(x), F] < \infty$, so together these show that $[A : F] < \infty$. But then $A$ is a finite division ring, so $A$ is a field by Wedderburn's theorem. This is a contradiction because $x$ and $y$ do not commute. $\square$

**Corollary 6.15.** *If $D$ is a division ring such that for any $x \in D$, we have $x^{n(x)} = x$ for some $n(x) \in \mathbb{N}$, then $D$ is commutative.*

*Proof.* Just note that $D$ must have positive characteristic because otherwise $2 \in D$, and $2^n \neq 2$ for any $n \in \mathbb{N}$. Thus $D$ is algebraic over its prime subfield, which is finite, ad the result follows from the theorem. $\square$

In fact we can generalize Jacobson's theorem to apply to algebraic algebras over finite fields which have no nonzero nilpotent elements. First we need a lemma, due to Andrunakievich, which says that a reduced ring is a subdirect product of domains.

**Definition 6.16.** A ring with no nonzero nilpotent elements is called *reduced*.

Note that a reduced ring is prime iff it's a domain. For if $R$ is reduced and $ab = 0$, then $(bRa)^2 = 0$, so every element of $bRa$ is nilpotent and thus $bRa = 0$. If $R$ is prime and $a \neq 0$, the left annihilator of the left ideal $Ra$ must be zero, so in particular $b = 0$.

We can push this a bit further. Let $R$ be a reduced ring, let $S$ be any subset of $R$, and suppose $aS = 0$. Then $(Sa)^2 = 0$ so $Sa = 0$. Hence $\operatorname{l.ann}(S) = \operatorname{r.ann}(S)$ is an ideal of $R$. Moreover, $R/\operatorname{l.ann}(S)$ is reduced. For if $a^2S = 0$, then $(aSa)^2 = 0 \implies aSa = 0 \implies (aS)^2 = 0 \implies aS = 0$.

**Theorem 6.17** (Andrunakievich)**.** *If $R$ is a reduced ring, then there is a collection of ideals $P_x \lhd R$ such that $\cap P_x = 0$, and each $R/P_x$ is a domain.*

*Proof.* Fix $x \in R$ and choose an ideal $P_x$ which is disjoint from $\{ x^i \mid i > 0 \}$ and maximal with respect to $R/P_x$ is reduced. We claim that $P_x$ is a prime ideal, so $R/P_x$ is a domain by the previous remark. If $P_x$ is not prime we can find ideals $A, B \lhd R$ with $A, B \supsetneq P$ and $AB \subseteq P$. But then $\overline{A} \in \operatorname{l.ann}_{\overline{R}}(\overline{B})$, so by the previous remarks, $R/A$ is reduced and so by our choice of $P_x$, $A$ must contain some power of $x$. Essentially the same argument shows that $B$ must contain some power of $x$ as well, and this is a contradiction because $AB \subseteq P_x$ which avoids $\{ x^i \mid i > 0 \}$.

To finish the proof, just note that $x \notin P_x$, so $\cap_{x \in R} P_x = 0$.

$\square$

**Theorem 6.18.** *An algebraic domain is a division ring.*

*Proof.* Let $R$ be a domain which is algebraic over $F$ and choose $0 \neq x \in R$. Then

$$a_n x^n + \ldots + a_1 x + a_0 = 0$$

and moreover $a_0 \neq 0$ because otherwise $x$ would be a zero divisor. Rearranging terms and dividing through by $a_0$, we get

$$1 = x(-1/a_0)(a_n x^{n-1} + \ldots + a_1)$$

so $x$ is a unit. $\square$

In fact, we've actually shown that in any ring, an element which is both algebraic and regular is invertible.

**Theorem 6.19.** *Let $F$ be a finite field, and $R$ an algebraic $F$-algebra with no nonzero nilpotent elements. Then $R$ is commutative.*

*Proof.* By Andrunakievich's Lemma, we can find a collection of ideals $I_\alpha \lhd R$ such that $\cap I_\alpha = 0$ and each $R/I_\alpha$ is a domain. There is a natural ring injection

$$R \to \prod R/I_\alpha$$

given by sending $r \in R$ to the element of $\prod R/I_\alpha$ whose $\alpha^{th}$ component is $r + I_\alpha$. This map is an injection because it's kernel is precisely $\cap I_\alpha = 0$. Thus it suffices to show that each $R/I_\alpha$ is commutative. But since $R$ is algebraic over $F$, so too is each $R/I_\alpha$, so we just need to show that a domain, algebraic over a finite field, is commutative. But an algebraic domain is a division ring, hence each $R/I_\alpha$ is a division ring, algebraic over a finite field, so is commutative by Jacobson's theorem. $\square$

## 7. Gelfand-Kirillov dimension

We turn now to a useful algebra invariant, first introduced by Gelfand and Kirillov in 1966 in order to distinguish certain kinds of division rings. Recall that an algebra $A$ over a field $F$ is called *affine* if $A$ is finitely generated as an $F$-algebra.

**Definition 7.1.** Let $A$ be an affine $F$-algebra, and let $V \subseteq A$ be a finite dimensional $F$ subspace (containing 1) which generates $A$ as algebra. Let $V^n$ denote the subspace spanned by all products of $n$ elements from $V$. There is a chain of subspaces

$$F \subseteq V \subseteq V^2 \subseteq \cdots \subseteq \cup_{n \geq 0} V^n = A$$

and we define the *Gelfand-Kirillov dimension* of $A$, denoted $\mathrm{GKdim}(A)$, by

$$\mathrm{GKdim}(A) = \limsup \frac{\log \dim(V^n)}{\log n}.$$

If $A$ is any $F$-algebra (not necessarily affine) then we define the Gelfand-Kirillov dimension of $A$ by

$$\mathrm{GKdim}(A) = \sup\{\, \mathrm{GKdim}(B) \mid B \text{ is an affine subalgebra of } A\,\}.$$

Although this definition seems to depend upon our particular choice of generating subspace $V$, in fact this is mere illusion. For suppose that $W$ is some other generating subspace (also assumed to contain 1). Then since $A = \cup_{n \geq 0} V^n$, there is some integer $s$ such that $W \subseteq V^s$. Hence $\dim(W^n) \leq \dim(V^{sn})$ and so

$$\frac{\log \dim(W^n)}{\log n} \leq \frac{\log \dim(V^{sn})}{\log sn} \cdot \frac{\log sn}{\log n}$$

and since $\frac{\log sn}{\log n} \to 1$ as $n \to \infty$, taking limsups shows that the Gelfand-Kirillov dimension computed via $W$ is at most the Gelfand-Kirillov dimension computed via $V$. By symmetry, these must in fact be equal.

**Example 7.2.**

- Finite dimensional algebras are precisely those affine algebras with Gelfand-Kirillov dimension zero. More generally, $\mathrm{GKdim}(A) = 0$ iff $A$ is *locally finite*, meaning that every affine subalgebra is finite dimensional.
- $\mathrm{GKdim}(F[x_1, \ldots, x_n]) = n$
- $\mathrm{GKdim}(F\langle x_1, \ldots, x_n\rangle) = \infty$ if $n \geq 2$
- Set $A = F\langle x, y\rangle/(y)^2$. Then $\mathrm{GKdim}(A) = 2$
- $A = \begin{pmatrix} F[x] & F[x,y] \\ 0 & F[y] \end{pmatrix}$. Then $\mathrm{GKdim}(A) = 2$. Note that $A$ is affine PI, but not noetherian on either side.

- GKdim($\mathbb{A}_1$) = 2, in fact, GKdim($\mathbb{A}_n$) = $2n$, where as usual $\mathbb{A}_n$ denotes the $n^{th}$ Weyl algebra.

We wish to show that a domain of finite Gelfand-Kirillov dimension is Ore, but first we will need some easy properties of GKdim.

**Proposition 7.3** (Properties of GKdim for affine algebras)**.**

(1) *If* $B \subseteq A$ *then* $\mathrm{GKdim}(B) \leq \mathrm{GKdim}(A)$.
(2) $\mathrm{GKdim}(A \oplus B) = \max\{\mathrm{GKdim}(A), \mathrm{GKdim}(B)\}$.
(3) *If* $I \lhd A$*, then* $\mathrm{GKdim}(A/I) \leq \mathrm{GKdim}(A)$.
(4) *If* $B \supseteq A$ *is such that* $B_A$ *is a finitely generated module, then* $\mathrm{GKdim}(B) = \mathrm{GKdim}(A)$.
(5) $\mathrm{GKdim}(M_n(A)) = \mathrm{GKdim}(A)$ *for every* $n$.
(6) *If* $I \lhd A$ *and* $I$ *contains a regular element, then* $\mathrm{GKdim}(A/I) \leq \mathrm{GKdim}(A) - 1$.
(7) $\mathrm{GKdim}(A[x]) = \mathrm{GKdim}(A) + 1$.
(8) $\mathrm{GKdim}(A \otimes_F B) \leq \mathrm{GKdim}(A) + \mathrm{GKdim}(B)$