On the Power of Random Greedy Algorithms

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Joint work with He Guo
(who created most of these slides)

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Context of this talk: Ramsey theory

A mysterious phenomenon

No matter how to partition a sufficiently large structure, there will always be a well-behaved substructure in one of the parts

- Some examples:

Large structure	**Substructure**	**Verified by**
Edges of K_n	Fixed graphs	Ramsey's theorem
$[N] = \{1, \ldots, N\}$	Arithmetic progressions	Van der Waerden's theorem

- More examples: Schur's, Erdős–Szekeres, Hales–Jewett theorem...

In this talk

- Present new lower bound on van der Warnden numbers
  - Partition $[N]$ into two parts avoiding specific APs
- Obtain such partition by a natural random greedy algorithm
  - Improve previous results that take all randomness at once

Van der Waerden number (Focus on 3-AP case)

Van der Waerden number $W(3,k)$

$W(3,k) :=$ minimum N such that every red/blue coloring of numbers in $[N] = \{1,\dots,N\}$ contains red 3-term arithmetic progression or blue k-AP

- $W(3,k) > N$: existence of a red/blue coloring of $[N]$ such that there is no red 3-AP or blue k-AP

Theorem (Brown–Landman–Robertson, 2007)

We have $W(3,k) = \Omega(k^2/k^{1/\log\log k})$.

Theorem (Li–Shu, 2008)

We have $W(3,k) = \Omega(k^2/(\log k)^2)$.

Theorem (Guo–Warnke, 2020+)

We have $W(3,k) = \Omega(k^2/\log k)$.

Proof strategy of previous results

Lower bound on van der Waerden number $W(3, k)$

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Their proof strategy: Take all randomness at once

- Color each number in $[N]$ by red & blue with prob. p & $1 - p$, resp.
- Lovász Local Lemma: $\mathbb{P}$(no red 3-AP or blue k-AP) > 0

- Li (2009): improve the log factor by 3-AP free process?

Proof strategy of our result

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- $W(3, k) > N$: existence of a set $I \subseteq [N]$ such that (i) I is 3-AP free, and (ii) $|I \cap K| \geq 1$ for all k-APs K

Theorem (Guo–Warnke, 2020+)

We have $W(3, k) = \Omega(k^2/\log k)$.

We construct such set $I \subseteq [N]$ by 3-AP free process

- Start with an empty set
- At each step, add one number uniformly at random, subject to the constraint that no 3-AP is created

$N = 9$ for example:

1, 2, 3, 4, 5, 6, 7, 8, 9

(Open numbers can be added. Closed numbers cannot.) $I = \varnothing$

Proof strategy of our result

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$N = 9$ for example:

$$\not{1},\ 2,\ 3,\ \not{4},\ 5,\ 6,\ \not{7},\ \not{8},\ 9$$

(Open numbers can be added. Closed numbers cannot.) $I = \{2,3,5\}$

Proof strategy of our result

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(Open numbers can be added. Closed numbers cannot.) $I = \{2, 3, 5, 9\}$

Feature 1: Only polynomially many k-APs in $[N]$

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- The set $I \subseteq [N]$ constructed by the 3-AP free process satisfies that
  (i) I is 3-AP free (by the definition of the process)
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The first feature: only polynomially many k-APs (for the union bound)

- The 3-AP free $I \subseteq [N]$ constructed by 3-AP free process has to satisfy

$$|I \cap K| \geq 1$$

  for all k-APs K in $[N]$, which are only $\Theta(N^2)$ many
- Exponentially many substructures in other Ramsey type problems

Feature 2: Only $O(1)$ many 3-APs containing two numbers

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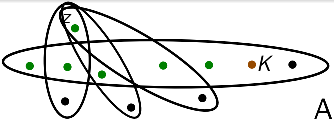
The second feature: only $O(1)$ many 3-APs containing two numbers

- One-step change of # open numbers in K is small
- Track it by concentration inequalities



Adding z can close some open numbers in K

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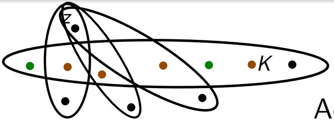

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Sketch of the proof

Theorem (Guo–Warnke, 2020+)

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Two features in 3-AP free process setting

- Only $\Theta(N^2)$ many k-APs in $[N]$
- We can track # open numbers in K throughout the process

Sketch of the proof

- At each step

$$\frac{\#\text{ open numbers in } K}{\#\text{ open numbers in } [N]} \approx \frac{k}{N} \quad \text{(Pseudo-randomness)}$$

- After m steps, where $km/N > 9 \log N$

$$\mathbb{P}(I \cap K = \varnothing) \approx \left(1 - \frac{k}{N}\right)^m \leq \exp(-km/N) \ll N^{-2}$$

A more general result: for all fixed $r \geq 2$

Van der Waerden number $W(r,k)$

$W(r,k) :=$ minimum N such that every red/blue coloring of numbers in $[N] = \{1, \dots, N\}$ contains red r-term arithmetic progression or blue k-AP

- $W(r,k) > N$: existence of a set $I \subseteq [N]$ such that (i) I is r-AP free, and (ii) $|I \cap K| \geq 1$ for all k-APs K

Theorem (Guo–Warnke, 2020+)

We have $W(r,k) = \Omega(k^{r-1}/(\log k)^{r-2})$ for fixed $r \geq 2$.

Proof idea: analyzing r-AP free process

Features in 3-AP free case carry over. Similar pseudo-random properties

- Improve Brown–Landman–Robertson (2007) & Li–Shu (2008) (LLL)
- Answer a question of Li from 2009

Open problems

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Theorem (Guo–Warnke, 2020+)

$W(r,k) = \Omega(k^{r-1}/(\log k)^{r-2})$ for fixed $r \geq 2$.

- $W(3,k) \leq \exp(k^{1/(1+\Omega(1))})$ Bloom–Sisask (2020)
- Fact: $W(3,k)$ grows like quadratically for $k = 1,2,\ldots,19$.

3	3	4	5	6	7	8	9	10	11	12	13	14
$W(3,k)$	9	18	22	32	46	58	77	97	114	135	160	186
k^2	9	16	25	36	49	64	81	100	121	144	169	196

Conjecture by Ahmeda–Kullmann–Snevily (2014)

$W(3,k) = O(k^2)$.

$250 Conjecture by Graham

$W(3,k) = O(f(k))$ for some polynomial function f.