Bounds on Ramsey Games via Alterations

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Joint work with He Guo

Context of this talk

Lower bound on Ramsey number $R(H, k)$

$R(H, k) > n$: need to show there exists an n-vertex graph G that is (i) H-free, and (ii) each k-vertex set contains at least one edge

Remarks:

- Probabilistic Method usually used to show existence of such G
- This problem inspired development of many important approaches:
  - Alteration method, Lovász local lemma, Semi-random, H-free process...

Topic of this talk

Refinement of Probabilistic Method approach for online Ramsey settings

Applications: New bounds for online Ramsey games

1. Ramsey, Paper, Scissors (extending Fox–He–Wigderson 2019+)
2. Online Ramsey numbers (extending Conlon–Fox–Grinshpun–He 2018)

History of studying lower bounds on Ramsey numbers

Lower bound on Ramsey number $R(H, k)$

$R(H, k) > n$: need to show there exists an n-vertex graph G that is
(i) H-free, and (ii) each k-vertex set contains at least one edge

- Testbed for new proof techniques/methods:

What kind of H?	**Authors**	**Methods**
K_3	Erdős (1961)	Alteration method
K_s or C_ℓ	Spencer (1975/77)	Lovász local lemma
Any graph H	Krivelevich (1995)	Alteration method
K_3	Kim (1995)	Semi-random
K_s or C_ℓ	Bohman–Keevash (2010)	H-free process
Many C_ℓ	Mubayi–Verstraëte (2019+)	Pseudo-random

- Online results by Conlon, Fox et al. only for K_3, based on Erdős (1961)
- Krivelevich's approach is not applicable for these online settings
- We refine alteration method and get online results for any graph H

Widely applications of Erdős and Krivelevich approaches

(Part of) Applications of Erdős (1961) or Krivelevich (1995) approaches

- Online Ramsey problems
  - Conlon et al. (2018); Fox et al. (2019+)
- Induced bipartite graph in triangle-free graphs
  - Erdős–Faudree–Pach–Spencer (1988); Kwan–Letzter–Sudakov–Tran (2018+); Batenburg–de Joannis de Verclos–Kang–Pirot (2018+); Guo–Warnke (2019++)
- Minimum induced tree in graphs
  - Erdős–Saks–Sós (1986)
- Various Ramsey numbers
  - Krivelevich (1998); Sudakov (2007)
- Erdős–Rogers function
  - Krivelevich (1995); Sudakov (2004)
- Coloring (hyper)graphs without certain structures
  - Osthus–Taraz (2000); Bohman–Frieze–Mubayi (2009)
- Ramsey–Turán problems

Review of alteration method by Krivelevich

Lower bound on Ramsey number $R(H, k)$

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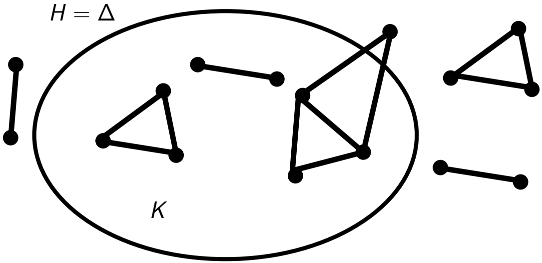
(i) H-free, and (ii) each k-vertex set contains at least one edge

High-level idea of Krivelevich (1995) for $n = \Theta\big((k/\log k)^{(e_H-1)/(v_H-2)}\big)$

Get $G \subseteq G_{n,p}$: remove edges of a maximal family of edge-disjoint H-copies

(i) G is H-free (otherwise the family can be increased)

(ii) Each k-vertex set K of G still contains ≥ 1 edge (main difficulty)



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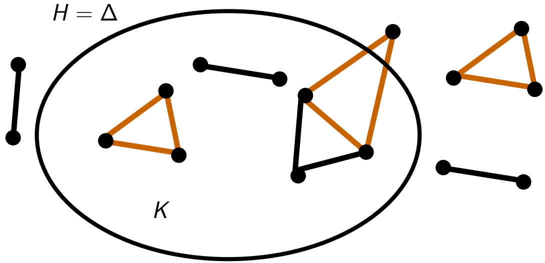

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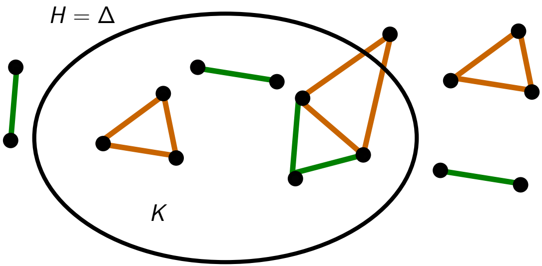

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(i) G is H-free (otherwise the family can be increased)

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Drawback: not applicable to design online algorithms

In online Ramsey games, players cannot foresee whether or not an existing edge will be contained in an H-copy in the future

$\longrightarrow$ We develop a variant that applies to online Ramsey settings

Our new approach: removing edges of ALL H-copies

High-level idea of Krivelevich

Get $G \subseteq G_{n,p}$: remove edges of a maximal family of edge-disjoint H-copies

(i) G is H-free (otherwise the family can be increased)

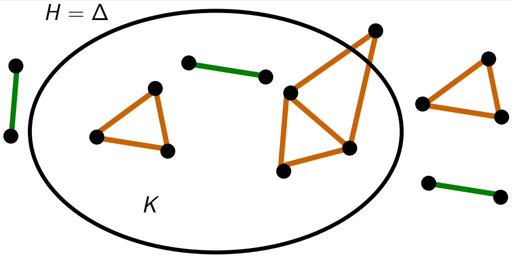
(ii) Each k-vertex set K of G still contains ≥ 1 edge

High-level idea of our new approach: removing ALL H-copies

Get $G \subseteq G_{n,p}$: remove edges of ALL H-copies

(i) G is H-free (all H-copies have been removed)

(ii) Each k-vertex set K of G still contains ≥ 1 edge (main difficulty)



Our new approach: what we need to show

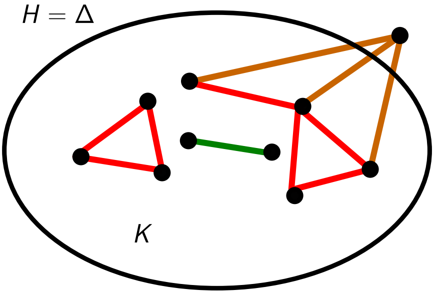
- Idea: remove edges of all H-copies in $G_{n,p}$ to get G
- Aim: show G satisfies
  - (i) n-vertex H-free (✓)
  - (ii) each k-vertex set K contains ≥ 1 edge

Need to show

$$|\{\text{edges of } G_{n,p} \text{ inside } K\}| > |\{\textit{removed edges inside } K\}|$$

By construction of G:

$$\{\textit{removed edges inside } K\} = \{\text{edges in } H\text{-copies of } G_{n,p} \text{ inside } K\}$$



Main technical result

- Idea: remove edges of all H-copies in $G_{n,p}$ to get G for $n = \Theta((\frac{k}{\log k})^{m_2(H)})$
- Aim: show G satisfies
  (i) n-vertex H-free (✓)
  (ii) each k-vertex set K contains ≥ 1 edge

Need to show

$$|\{\text{edges of } G_{n,p} \text{ inside } K\}| > |\{\textit{removed edges inside } K\}|$$

By construction of G:

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Theorem (Guo–Warnke 2019+)

For any well-behaved H and $\delta > 0$, for all large $C_{\delta,H}$ and small $c_{\delta,H,C} > 0$,

$$\textit{if} \qquad n := \lfloor c(k/\log k)^{m_2(H)} \rfloor \qquad \textit{and} \qquad p := C(\log k)/k,$$

then in $G_{n,p}$ whp (with high probability) for all k-vertex sets K:

$$|\{\textit{edges of } G_{n,p} \textit{ inside } K\}| \geq (1-\delta)\cdot \binom{k}{2} p,$$

$$|\{\textit{edges in } H\textit{-copies of } G_{n,p} \textit{ inside } K\}| \leq \delta \cdot \binom{k}{2} p.$$

Main technical result

- Idea: remove edges of all H-copies in $G_{n,p}$ to get G for $n = \Theta((\frac{k}{\log k})^{m_2(H)})$
- Aim: show G satisfies
  - (i) n-vertex H-free ($\checkmark$)
  - (ii) each k-vertex set K contains ≥ 1 edge ($\checkmark$)

Theorem (Guo–Warnke 2019+)

For any well-behaved H and $\delta > 0$, for all large $C_{\delta,H}$ and small $c_{\delta,H,C} > 0$,

$$\textit{if} \qquad n := \lfloor c(k/\log k)^{m_2(H)} \rfloor \qquad \textit{and} \qquad p := C(\log k)/k,$$

then in $G_{n,p}$ whp (with high probability) for all k-vertex sets K:

$$X_K := |\{\textit{edges of } G_{n,p} \textit{ inside } K\}| \geq (1-\delta)\cdot\binom{k}{2}p,$$

$$Y_K := |\{\textit{edges in } H\textit{-copies of } G_{n,p} \textit{ inside } K\}| \leq \delta\cdot\binom{k}{2}p.$$

How to verify (ii)?

- $|E(G[K])| = X_K - Y_K \geq (1-2\delta)\cdot\binom{k}{2}p \geq 1$ for $\delta < 1/2$ ($\checkmark$)

Task: $Y_K = \#$ edges in H-copies of $G_{n,p}$ inside $K \le \delta\binom{k}{2}p$

- Aim: n-vertex H-free graph G without independent set of size k
- Strategy: Remove all edges of all H-copies in $G_{n,p}$ to get G, where

$$n \sim c\left(\frac{k}{\log k}\right)^{m_2(H)} \qquad \text{and} \qquad p = C\frac{\log k}{k}$$

Simple bound fails, where $\mathcal{H}_K := \{H\text{-copies in } G_{n,p} \text{ with } \ge 1 \text{ edge inside } K\}$

Trivial bound: $Y_K \le e_H|\mathcal{H}_K|$, but $\mathbb{P}(|\mathcal{H}_K| \ge \varepsilon\binom{k}{2}p) \gg \binom{n}{k}^{-1}$

- Rules out naive union bound over all k-vertex sets K in $G_{n,p}$

"Infamous" upper tail behavior: example $H = K_s$

1. For $t := \Theta\left((\varepsilon\binom{k}{2}p)^{\frac{1}{s}}\right)$, one K_t contains $\Theta(t^s) \ge \varepsilon\binom{k}{2}p$ many K_s-copies
2. As $t \ll k$, one K_t fits inside K. Then

$$\mathbb{P}(|\mathcal{H}_K| \ge \varepsilon\binom{k}{2}p) \ge \mathbb{P}(\text{one } K_t \text{ occurs in } G_{n,p}[K]) \ge p^{\binom{t}{2}} \gg \binom{n}{k}^{-1}$$

Main message

Must handle H-copies that share a common edge inside K more carefully

Task: $Y_K = \#$ edges in H-copies of $G_{n,p}$ inside $K \le \delta\binom{k}{2}p$

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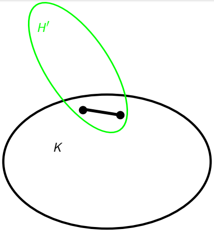
Probabilistic & Combinatorial Observations

1. (**P**) Main contribution to Y_K: H-copies with exact 2 vertices inside K
   - "good" copies; $\#$"bad" copies is negligible (as $n \gg k$)
2. (**C**) Multiple good copies on one edge contribute 1 to Y_K
   - Select one representative H-copy for each such edge
3. (**P**) $|\{\text{representative good copies}\}| \approx |\text{max. edge-disjoint subfamily}|$
   - $\#$ pairs of intersecting representative good copies is negligible



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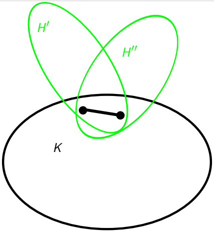

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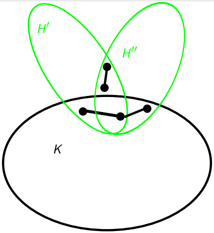

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3. (**P**) $|\{\text{representative good copies}\}| \approx |\text{max. edge-disjoint subfamily}|$
   - $\#$ pairs of intersecting representative good copies is negligible
   - $\mathcal{I}_K :=$ this subfamily of edge-disjoint good H-copies

Combining these three observations, we have

$$Y_K \approx |\mathcal{I}_K|.$$

Intuitively, $|\mathcal{I}_K|$ behaves like sum of independent random variables:

$$\mathbb{P}\Big(|\mathcal{I}_K| \ge \delta\tbinom{k}{2}p/2\Big) \le \exp\Big(-\Theta(\delta\tbinom{k}{2}p)\Big) \ll n^{-k}, \text{ by } C \ge C_0(\delta, H)$$

A brief summery

H is well-behaved: strictly 2-balanced graph with $m_2(H) > 1$

Theorem (Guo–Warnke 2019+)

For any well-behaved H and $\delta > 0$, for all large $C_{\delta,H}$ and small $c_{\delta,H,C} > 0$,

$$\text{if} \qquad n := \lfloor c(k/\log k)^{m_2(H)} \rfloor \qquad \text{and} \qquad p := C(\log k)/k,$$

then in $G_{n,p}$ whp for all k-vertex sets K:

$$\#\text{ edges in } H\text{-copies of } G_{n,p} \text{ inside } K \leq \delta \cdot \binom{k}{2} p.$$

Corollary (Random-like n-vertex H-free graph G without ISET of size k)

Remove all edges of all H-copies in $G_{n,p}$ (as above) to get G. Whp for all k-vertex sets K:

$$|E(G[K])| = (1 \pm \varepsilon) \cdot \binom{k}{2} p.$$

- *G is random-like, e.g., $\deg_G(v) = (1 \pm \varepsilon) \cdot np$, $|E(G)| = (1 \pm \varepsilon) \cdot \binom{n}{2} p$*

Motivation: Online Ramsey games

Lower bound on Ramsey number $R(H, k)$

$R(H, k) > n$: need to show there exists an n-vertex graph G that is
(i) H-free, and (ii) each k-vertex set contains at least one edge

- Polynomial gaps from best upper bounds, despite long history

What kind of H?	**Authors**	**Methods**
K_3	Erdős (1961)	Alteration method
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K_3	Kim (1995)	Semi-random
K_s or C_ℓ	Bohman–Keevash (2010)	H-free process
Many C_ℓ	Mubayi–Verstraëte (2019+)	Pseudo-random

Motivation for Online Ramsey settings

- Development of new proof techniques

Application 1: Ramsey, Paper, Scissors (RPS)

Ramsey, Paper, Scissors game

- Board: n initially isolated vertices; Two players: **Proposer**, **Decider**
- In each turn, *simultaneously*:
  - **Proposer** proposes a new pair in $\binom{[n]}{2}$ that does not form an H-copy with edges of current graph
  - **Decider** decides it to be an edge or not (without knowing the pair)
- **Proposer** wins if in the final graph $\exists$ ISET of size k

Ramsey, Paper, Scissors number $RPS(H, n)$

$RPS(H, n) =$ maximum k: **Proposer** has a strategy to win with prob. $\geq \frac{1}{2}$

Theorem (Fox–He–Wigderson 2019+)

$RPS(K_3, n) = \Theta(n^{1/2} \log n)$.

- Their upper bound is based on Erdős (1961) construction for $R(K_3, k)$

Theorem (Guo–Warnke 2019+)

$RPS(H, n) = O(n^{1/m_2(H)} \log n)$ *for any well-behaved* H.

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- **Proposer** wins if in the final graph $\exists$ ISET of size k

Theorem (Guo–Warnke 2019+: Ramsey, Paper, Scissors)

No matter how Proposer plays, Decider has a randomized strategy so that whp there is no ISET of size $k = \Theta(n^{1/m_2(H)} \log n)$ in the final graph.

Analysis: Decider adds each pair as edge independently with probability p

1. $\binom{[n]}{2} = \{\text{proposed pairs}\} \sqcup \{\text{pairs not proposed}\}$ at the end of the game
2. Final graph $G \subseteq G_{n,p}$: $E(G) = p$-random subset of proposed pairs
3. Not proposed pair forms H-copy with $E(G) \Rightarrow |E(G[K])| \geq X_K - Y_K \geq 1$

Application 2: Online Ramsey numbers

(H, k)-online Ramsey game

- Board: infinite initially isolated vertices; Two players: **Builder**, **Painter**
- Each turn: **Builder** builds a new edge, *then* **Painter** paints it red/blue
- **Builder** wins if a red H or blue K_k shows up

Online Ramsey number $\tilde{r}(H, k)$

$\tilde{r}(H, k) =$ minimum N: **Builder** has a strategy to win for sure in N turns

Theorem (Conlon–Fox–Grinshpun–He 2018)

$\tilde{r}(K_3, k) = \Omega\big(k \cdot (\frac{k}{\log k})^2\big)$ + *many further results.*

- Their lower bound is based on Erdős (1961) construction for $R(K_3, k)$
  - Erdős (1961) really uses special structure of K_3

Theorem (Guo–Warnke 2019+)

$\tilde{r}(H, k) = \Omega\big(k \cdot (\frac{k}{\log k})^{m_2(H)}\big)$ *for any graph H.*

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Theorem (Guo–Warnke 2019+: Online Ramsey numbers)

No matter how Builder builds an edge in each turn, Painter has a strategy to paint it red/blue to avoid red H *or blue* K_k *in* $\Omega\big(k \cdot (\frac{k}{\log k})^{m_2(H)}\big)$ *turns.*

- Infinite vertex-set board causes difficulties for taking union bound

Remarks on the applications to online Ramsey games

Theorem (Guo–Warnke 2019+: Ramsey, Paper, Scissors)

No matter how Proposer plays, Decider has a randomized strategy so that whp there is no ISET of size $k = \Theta(n^{1/m_2(H)} \log n)$ in the final graph.

Theorem (Guo–Warnke 2019+: Online Ramsey numbers)

No matter how Builder builds an edge in each turn, Painter has a strategy to paint it red/blue to avoid red H or blue K_k in $\Omega\big(k \cdot (\frac{k}{\log k})^{m_2(H)}\big)$ turns.

- Results of Fox, Conlon et al. based on Erdős lower bound on $R(K_3, k)$
    - Erdős (1961) construction really uses special structure of K_3
- Krivelevich's approach is not applicable to design online algorithms
    - players cannot foresee creation of H-copies on existing edges
- We have randomized algorithms for Painter/Decider
    - Based on our approach to $R(H, k)$; Benefit from deleting all H-copies

Summary

Main point of our new approach

After removing all H-copies from $G_{n,p}$, the remaining G is still random-like

Remarks:

- Previous approaches only remove some H-copies
- Advantage of our refined approach: works in online settings

Two applications:

1. Ramsey, Paper, Scissors (extending Fox et al. 2019+)
2. Online Ramsey numbers (extending Conlon et al. 2018)